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ASYMPTOTICALLY STABLE DYNAMIC RISK ASSESSMENTS

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Abstract

In this paper asymptotically stable risk assessments are studied. They are characterized by not being sensitive with respect to huge additional capital in the very far future. Under the additional hypothesis of being locally continuous from below, these risk assessments are exactly those which allow a robust representation with so-called local test probabilities having a support with finite time horizon.

Time-consistent risk assessments can be constructed by composing a sequence of generators. We give several conditions for the generators such that the resulting risk assessments are indeed asymptotically stable.

Key words and phrases: *asymptotic stability of risk assessments, construction by generators, local test probabilities, robust representation, time-consistency.*

AMS - Classification: *60G35, 91B30, 91B16*

1. Introduction

The starting point of this paper is the investigation of risk assessments which are not influenced by statements like: "Far in the future I shall be extremely rich". Such assessments are called asymptotically stable.

Risk assessments or its negative notation, the risk measures, have been widely studied since the lighthouse paper of Artzner et al. in 1999 (see [2]). For a good list of articles about risk assessments we refer to [1]. While first simple or dynamic risk assessments of random variables have been studied, the focus of the investigations is now the dynamic assessments of processes either with discret or continuous time space with finite or infinite time horizon, see for instance [3], [4], [5], [14], [15], [6], [1], [12],[10].

In the first main result of this paper we characterize asymptotic stable risk assessments under the additional assumption of local continuity from below by a robust representation. It

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turns out that the test probabilities of this representation are exactly the so-called local probabilities whose support is restricted to a finite time horizon. This is equivalent to the fact that the acceptance set of the risk assessment is closed in the weak topology created by local test probabilities.

Using the property of time-consistency, we show in the second part of the paper how such asymptotically stable risk assessments can be constructed out of a time sequence of generators which have to satisfy a corresponding property of asymptotic stability. We also investigate the case where the generators at time s are functionals of the next cumulative value X_{s+1} and the future increments whose influence is measured by a series of deterministic functions. A condition for these functions is given which yields asymptotically stable risk assessments.

The structure of this paper is as follows. In Section 2 we give the main notation and definitions. The robust representation of asymptotically stable risk assessments is presented in the Section 3. In Section 4 we construct dynamic risk assessments which are asymptotically safe and finally give some examples in Section 5.

2. Preliminaries

Let $\mathbb{N} := \{1, 2, \dots\}$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space. All relations between random variables, stochastic processes or sets are understood to hold $\mathbb{P}$-almost surely. As usual, L^p (resp. L^p_t) is the Banach space of $\mathcal{F}$-measurable (resp. $\mathcal{F}_t$-measurable) random variables Z with finite $\|.\|_p$-norm, where $\|Z\|_p := (\mathbb{E}[|Z|^p])^{1/p}$ for $p < \infty$ and $\|Z\|_\infty := \text{ess.sup}\,|Z|$. By $\mathcal{R}^\infty$ we denote the linear space of adapted processes $X : \Omega \times \mathbb{N} \to \mathbb{R}$ such that $X^* := \sup_{t\in\mathbb{N}} |X_t| \in L^\infty$, with the partial order $X \geq Y$ whenever $X_t \geq Y_t$ for all $t \in \mathbb{N}$. Let $\mathcal{A}^1$ be the linear space of adapted processes $a : \Omega \times \mathbb{N} \to \mathbb{R}$ such that $\sum_{t\in\mathbb{N}} |\Delta a_t| \in L^1$, where $\Delta a_t := a_t - a_{t-1}$ with the convention $a_0 = 0$. Further, $\mathcal{A}^1_+$ denotes the set of those $a \in \mathcal{A}^1$ which are nondecreasing, $\mathcal{A}^1_1$ the set of those $a \in \mathcal{A}^1_+$ for which $\mathbb{E}\left[\sum_{t\in\mathbb{N}} \Delta a_t\right] = 1$ and $\mathcal{A}^{1,\text{loc}}$ the set of those $a \in \mathcal{A}^1$ which are eventually constant, that is $a_t = a_T$ for all $t \geq T$ for some time horizon $T \in \mathbb{N}$. We set $\mathcal{A}^{1,\text{loc}}_1 := \mathcal{A}^{1,\text{loc}} \cap \mathcal{A}^1_1$. Elements of $\mathcal{A}^{1,\text{loc}}_1$ are regarded as *local test probabilities*.

The linear spaces $\mathcal{R}^\infty$ and $\mathcal{A}^1$ are in duality by the dual pairing

$$\langle X, a \rangle := \mathbb{E}\left[\sum_{t\in\mathbb{N}} X_t \Delta a_t\right].$$

For $a \in \mathcal{A}^{1,\text{loc}}$ one has $\langle X, a \rangle := \sum_{t=1}^{T} \mathbb{E}\left[X_t \Delta a_t\right]$ for some $T \in \mathbb{N}$.

Definition 2.1. A concave risk assessment on $\mathcal{R}^\infty$ is a function $\phi : \mathcal{R}^\infty \to \mathbb{R}$, which satisfies for all $X \in \mathcal{R}^\infty$

(i) $\phi(0) = 0$,

(ii) $\phi(X + m) = \phi(X) + m$ for all $m \in \mathbb{R}$,

(iii) $\phi(X) \geq \phi(Y)$ whenever $X \geq Y$,

(iv) $\phi(\lambda X + (1 - \lambda)Y) \geq \lambda\phi(X) + (1 - \lambda)\phi(Y)$ for all $\lambda \in (0, 1)$.

If in addition to (i) – (iv), ϕ is positively homogeneous, i.e. $\phi(\lambda X) = \lambda\phi(X)$ for all $\lambda \geq 0$ then ϕ is called coherent.

The acceptance set of a risk assessment ϕ is defined as $\mathcal{C} := \{X \in \mathcal{R}^\infty : \phi(X) \geq 0\}$. For every time horizon $T \in \mathbb{N}$ we define

$$\mathcal{R}_T^\infty := \{X \in \mathcal{R}^\infty : X_t = X_T \text{ for all } t \geq T\}, \quad \text{and similarly} \tag{2.1}$$

$$A_T^1 := \{a \in \mathcal{A}^1 : a_t = a_T \text{ for all } t \geq T\}. \tag{2.2}$$

Definition 2.2. A risk assessment $\phi : \mathcal{R}^\infty \to \mathbb{R}$ is locally continuous from below, if

$$\phi\left(X^n \mathbb{1}_{(0,T)} + X_T^n \mathbb{1}_{[T,\infty)}\right) \to \phi\left(X \mathbb{1}_{(0,T)} + X_T \mathbb{1}_{[T,\infty)}\right).$$

for all $T \in \mathbb{N}$ and every sequence (X^n) in $\mathcal{R}_T^\infty$ which increases to some $X \in \mathcal{R}_T^\infty$.

3. Asymptotically Stable Risk Assessments

In this paper we characterize risk assessments on $\mathcal{R}^\infty$, which satisfy one of the following properties:

(A1) If $X \notin \mathcal{C}$ then there exists a time horizon $T \in \mathbb{N}$ such that $X\mathbb{1}_{(0,T)} + N\mathbb{1}_{[T,\infty)} \notin \mathcal{C}$ for all $N \in \mathbb{N}$.

(A2) $\gamma_t(X) := \sup_{N \in \mathbb{N}} \left[\phi(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}) - \phi(X)\right] \to 0$ as $t \to \infty$.

Notice that (A1) is equivalent to the following statement: $X \in \mathcal{C}$ if and only if for any $t \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that $X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)} \in \mathcal{C}$. Of course, the "only if" part of the statement is the important one.
Risk assessments which satisfy (A1) have the desired property that an unacceptable position cannot become acceptable by adding a huge cash-flow far in the future. Even though such risk assessments neglect asymptotic benefits, they may take into account asymptotic losses.

Proposition 3.1. *The conditions (A1) and (A2) are equivalent.*

Proof. Suppose that (A1) holds, but $\sup_{t\in\mathbb{N}} \gamma_t(X) \geq \varepsilon$ for some $\varepsilon > 0$. By the translation property (ii) of the risk assessment we may assume that $\phi(X) = -\varepsilon/2$ so that $X \notin \mathcal{C}$. Hence, for any $t \in \mathbb{N}$ one has

$$\sup_{N\in\mathbb{N}} \phi(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}) = \gamma_t(X) + \phi(X) \geq \frac{\varepsilon}{2},$$

so that $X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)} \in \mathcal{C}$ for all $t \in \mathbb{N}$ and sufficiently large $N \in \mathbb{N}$. Property (A1) yields $X \in \mathcal{C}$, which is a contradiction. Hence (A2) has to hold.

Conversely, suppose that (A2) holds and $X \notin \mathcal{C}$. Since

$$\sup_{N\in\mathbb{N}} \phi(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}) = \gamma_t(X) + \phi(X), \quad \text{for all } t \in \mathbb{N},$$

$\phi(X) < 0$ and $\gamma_t(X) \to 0$, it follows that $\sup_{N\in\mathbb{N}} \phi(X\mathbb{1}_{(0,t_0)} + N\mathbb{1}_{[t_0,\infty)}) < 0$ for some t_0 large enough. Hence $X\mathbb{1}_{(0,t_0)} + N\mathbb{1}_{(t_0,\infty)} \notin \mathcal{C}$ for all $N \in \mathbb{N}$, which shows (A1). □

The conditions (A1) or (A2) leads us to the following definition:

Definition 3.1. A concave risk assessment ϕ satisfying one of the equivalent conditions of (A1) or (A2) is called *asymptotically stable*.

Generally speaking, the theory of robust representations of risk assessments yields with formulas of the form

$$\phi(X) = \inf_{a\in\mathscr{A}} \{\langle X, a\rangle - \phi^*(a)\} \quad \text{for all } X, \tag{3.1}$$

where the 'dual set' $\mathscr{A}$ is a convex set of linear forms on the space where ϕ is defined, and the conjugate function ϕ^* is given by

$$\phi^*(a) := \inf_{X\in\mathcal{R}^\infty} \{\langle X, a\rangle - \phi(X)\} = \inf_{X\in\mathcal{C}} \langle X, a\rangle \quad \text{for all } a \in \mathscr{A}, \tag{3.2}$$

and takes values in $[-\infty, 0]$. The second equality in (3.2) is a consequence of the following series of inequalities:

$$\inf_{X\in\mathcal{C}} \langle X, a\rangle \geq \inf_{X\in\mathcal{C}} \{\langle X - \phi(X), a\rangle\} \geq \inf_{X\in\mathcal{R}^\infty} \{\langle X - \phi(X), a\rangle\} \geq \inf_{X'\in\mathcal{C}} \langle X', a\rangle$$

because $X - \phi(X) \in \mathcal{C}$.
In our context, the robust representation of ϕ is given in the following theorem which is our main result in the static case. It characterizes the property of asymptotical stability for risk assessments which are locally continuous from below via the applications of local test probabilities.

Theorem 3.2. *Let $\phi : \mathcal{R}^\infty \to \mathbb{R}$ be a concave risk assessment which is locally continuous from below. The following statements are equivalent:*

(i) *ϕ is asymptotically stable.*

(ii) *The acceptance set $\mathcal{C}$ is $\sigma\big(\mathcal{R}^\infty, \mathcal{A}^{1,\mathrm{loc}}\big)$-closed.*

(iii) *ϕ is $\sigma\big(\mathcal{R}^\infty, \mathcal{A}^{1,\mathrm{loc}}\big)$-upper semicontinuous.*

(iv) *ϕ has a robust representation with local test probabilities:*

$$\phi(X) = \inf_{a \in \mathcal{A}_1^{1,\mathrm{loc}}} \{\langle X, a\rangle - \phi^*(a)\} \tag{3.3}$$

for all $X \in \mathcal{R}^\infty$.

(v) *For any sequence (X^n) and X in $\mathcal{R}^\infty$ such that X_t^n is bounded and $X_t^n \to X_t$ $\mathbb{P}$-almost surely for all $t \in \mathbb{N}$, one has $\phi(X) \geq \limsup_{n\to\infty} \phi(X^n)$.*

Proof.
$(i) \Rightarrow (ii)$: We have to show that

$$X \in \mathcal{C} \quad \Longleftrightarrow \quad \langle X, a\rangle - \phi^*(a) \geq 0 \quad \text{for all } a \in \mathcal{A}_1^{1,\mathrm{loc}}. \tag{3.4}$$

That the left hand side implies the right hand side follows directly from the definition of ϕ^*. The converse direction is more complicated and shown in the following six steps.
Step 1: If $X \notin \mathcal{C}$ then by the translation property there exists $\varepsilon \in (0,1]$ with $X + \varepsilon \notin \mathcal{C}$, such that by (A1) there exists $t \in \mathbb{N}$ such that

$$(X+\varepsilon)\mathbb{1}_{(0,t)} + (N+1)\mathbb{1}_{[t,\infty)} \notin \mathcal{C} \quad \text{for all } N \in \mathbb{N}. \tag{3.5}$$

Step 2: Here we make a deviation to use the duality between $L^\infty(\Omega', \mathcal{F}')$ and the set $\mathcal{M}^f(\Omega', \mathcal{F}')$ of finitely additive measures on some measurable space $(\Omega', \mathcal{F}')$. We follow [3], [5], [12], or [1].
Set $\Omega' := \Omega \times \mathbb{N}$, $\mathcal{F}' := \sigma\{\mathcal{F}_n \times \{n\} : n \in \mathbb{N}\}$, $\mu^0(B) := \sum_{n\geq 1} 2^{-n}\mathbb{P}(B_{|n})$ where $B_{|n} = \{\omega \in \Omega : (\omega, n) \in B\}$ for $B \in \mathcal{F}'$. μ^0 is a probability measure on $(\Omega', \mathcal{F}')$.
Further, let $\mathcal{M}_1^f$ the set of positive finitely additive measures μ on $(\Omega', \mathcal{F}')$ with $\mu(\Omega') = 1$ and $\mu(B \times \{n\}) = 0$ for all $B \in \mathcal{F}_n$ with $\mathbb{P}[B] = 0$. Identifying $X \in \mathcal{R}^\infty$ with $X'(\omega, n) = X_n(\omega) \in L^\infty(\Omega', \mathcal{F}')$ and writing simply $\langle X, \mu\rangle$ instead of $\langle X', \mu\rangle$, we get for $\phi'(X') = \phi(X)$ the representation (see [13])

$$\phi(X) = \phi'(X') = \min_{\mu \in \mathcal{M}_1^f} \{\langle X, \mu\rangle - \phi'^*(\mu)\}.$$

where again $\phi'^*(\mu) := \inf_{X\in\mathcal{C}} \langle X', \mu\rangle$ takes values in $[-\infty, 0]$.
Now the statement (3.5) implies the existence of a sequence μ^N in $\mathcal{M}_1^f$ with

$$\Big\langle (X+\varepsilon)\mathbb{1}_{(0,t)} + (N+1)\mathbb{1}_{[t,\infty)}, \mu^N \Big\rangle - \phi'^*(\mu^N) < 0 \quad \text{for all } N \in \mathbb{N}. \tag{3.6}$$

Since $\phi'^*(\mu^N) \leq 0$, it follows that

$$\left\langle \mathbb{1}_{\Omega\times[t,\infty)}, \mu^N \right\rangle \leq \frac{\|X\|_{\mathcal{R}^\infty} + \varepsilon}{N+1} \quad \text{and} \quad 0 \geq \phi'^*(\mu^N) \geq -\|X\|_{\mathcal{R}^\infty} \quad \text{for all } N \in \mathbb{N}. \tag{3.7}$$

Step 3: Next we show that the sequence (μ^N) satisfies:

$$\begin{aligned} &\forall\, \eta > 0, \forall\, B_\kappa \in \mathcal{F}', B_\kappa \nearrow \Omega' \text{ for } \kappa \to \infty\ \exists\, N_0, \kappa_0 \in \mathbb{N}, \text{ such that} \\ &\left\langle \mathbb{1}_{B_\kappa}, \mu^N \right\rangle \geq 1 - \eta \text{ for all } N \geq N_0 \text{ and } \kappa \geq \kappa_0. \end{aligned} \tag{3.8}$$

Let $\eta > 0$ and $B_\kappa \in \mathcal{F}'$ be an increasing sequence of subsets of Ω' with $B_\kappa \nearrow \Omega'$ for $\kappa \to \infty$. By (3.7) we find $N_0 \in \mathbb{N}$ such that $\left\langle \mathbb{1}_{\Omega\times[t,\infty)}, \mu^N \right\rangle \leq \eta$ for all $N \geq N_0$ and we take $M \in \mathbb{N}$ with $M \geq \|X\|_{\mathcal{R}^\infty}/\eta$. By the local continuity from below of ϕ we get

$$\begin{aligned} &\lim_{\kappa\to\infty} \inf_{N\geq N_0} \left\{ \left\langle M\mathbb{1}_{B_\kappa}, \mu^N \right\rangle - \phi'^*(\mu^N) \right\} \\ &\qquad \geq \lim_{\kappa\to\infty} \inf_{N\geq N_0} \left\{ \left\langle M(\mathbb{1}_{B_\kappa\cap\Omega\times(0,t)} + \mathbb{1}_{\Omega\times[t,\infty)}), \mu^N \right\rangle - \phi'^*(\mu^N) \right\} - M\eta \\ &\qquad \geq \lim_{\kappa\to\infty} \phi'(M(\mathbb{1}_{B_\kappa\cap\Omega\times(0,t)} + \mathbb{1}_{\Omega\times[t,\infty)})) - M\eta \\ &\qquad = \lim_{\kappa\to\infty} \phi\left(M(\mathbb{1}_{B_\kappa\cap\Omega\times\{1\}}, \ldots, \mathbb{1}_{B_\kappa\cap\Omega\times\{t-1\}}, \mathbb{1}_{\Omega\times\{t\}}, \ldots)\right) - M\eta \\ &\qquad = \phi(M\mathbb{1}_{(0,\infty)}) - M\eta = M(1-\eta). \end{aligned}$$

With the second inequality in (3.7), it follows that

$$1 \geq \lim_{\kappa\to\infty} \inf_{N\geq N_0} \left\langle \mathbb{1}_{B_\kappa}, \mu^N \right\rangle \geq 1 - \eta - \|X\|_{\mathcal{R}^\infty}/M \geq 1 - 2\eta$$

This shows (3.8).

Step 4: In this step we prove

$$\begin{aligned} &\forall\, \eta > 0,\ \exists\, \delta > 0 \text{ and } N \in \mathbb{N}, \text{ so that } \forall D \in \mathcal{F}' \text{ with } \sup_{0\leq N'\leq N} \left\langle \mathbb{1}_D, \mu^{N'} \right\rangle < \delta \\ &\text{implies } \left\langle \mathbb{1}_D, \mu^M \right\rangle < \eta \text{ for all } M \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Assume that (3.9) does not hold. Then there exists $\eta > 0$ such that for all $\delta_N = \eta/2^N$ there exists $D_N \in \mathcal{F}'$ with $\sup_{0\leq N'\leq N} \left\langle \mathbb{1}_{D_N}, \mu^{N'} \right\rangle < \eta/2^N$, but $\left\langle \mathbb{1}_{D_N}, \mu^M \right\rangle \geq \eta$ for some $M \in \mathbb{N}$. We set $B_\kappa := \Omega' \setminus \bigcup_{N\geq\kappa} D_N$ such that $\mu^0(B_\kappa) \geq 1 - 2\eta/2^\kappa$ or $B_\kappa \nearrow \Omega'$ for $\kappa \to \infty$. (Here we use the fact that μ^0 is σ-additive, while μ^N are only finitely additive for for $N \geq 1$). By (3.8) there exist $N_0, \kappa_0 \in \mathbb{N}$ such that $\left\langle \mathbb{1}_{B_\kappa}, \mu^M \right\rangle \geq 1 - \eta/2$ for all $M \geq N_0$ and $\kappa \geq \kappa_0$. Now for $\kappa' = \max(N_0, \kappa_0)$ we find not only $\sup_{0\leq N'\leq N} \left\langle \mathbb{1}_{B'_{\kappa'}}, \mu^{N'} \right\rangle < \eta/2^{\kappa'} \leq \eta/2$, but also $\left\langle \mathbb{1}_{B'_{\kappa'}}, \mu^M \right\rangle \leq \eta/2$ for all $M \geq \kappa'$, contradicting the fact that we have $\left\langle \mathbb{1}_{B'_{\kappa'}}, \mu^{M_{\kappa'}} \right\rangle \geq \eta$ for $M_{\kappa'} \in \mathbb{N}$. Thus assertion (3.9) is shown.

Step 5: We define $\overline{\mu} := \sum_{N'\geq 0} \mu^{N'}/2^{N'}$ and conclude for all $\eta > 0$ using $\delta > 0$ and N from (3.9): For any $B' \in \mathcal{F}'$ with $\langle \mathbb{1}_{B'}, \overline{\mu} \rangle < \delta/2^N$ we have $\left\langle \mathbb{1}_{B'}, \mu^{N'} \right\rangle < \delta$ for all $0 \leq N' \leq N$ such that $\left\langle \mathbb{1}_{B'}, \mu^M \right\rangle < \eta$ for all $M \in \mathbb{N}$, i.e.

$$\lim_{\langle \mathbb{1}_{B'}, \overline{\mu}\rangle \to 0} \left\langle \mathbb{1}_{B'}, \mu^M \right\rangle \to 0 \text{ uniformly for all } M. \tag{3.10}$$

By Theorem IV.9.12 in [9], the sequence (μ^M) is weakly sequentially compact and there exists a subsequence of (μ^N) (again denoted by (μ^N)) such that (μ^N) converges weakly to some $\widetilde{\mu} \in \mathcal{M}_1^f$.

Step 6: First (3.7) shows that $\left\langle \mathbb{1}_{\Omega\times[t,\infty)}, \widetilde{\mu}\right\rangle = 0$ and from (3.8) we conclude that for any $\eta > 0$ and any sequence $B_\kappa \in \mathcal{F}', B_\kappa \nearrow \Omega'$ for $\kappa \to \infty$ we have $\langle \mathbb{1}_{B_\kappa}, \widetilde{\mu}\rangle \geq 1-\eta$ for all sufficintly large κ. Therefore $\widetilde{\mu}$ is a probability measure absolutely continuous w.r.t. μ^0. Moreover, for every $\varepsilon' > 0$ there exists $Y \in \mathcal{C}'$ with

$$\phi'^*(\widetilde{\mu}) \geq \langle Y, \widetilde{\mu}\rangle - \varepsilon' = \lim_{N\to\infty} \left\langle Y, \mu^N \right\rangle - \varepsilon' \geq \liminf_{N\to\infty} \phi'^*(\mu^N) - \varepsilon'.$$

From (3.6) we conclude that

$$\left\langle X, \mu^N \right\rangle - \phi'^*(\mu^N) < -\varepsilon$$

for all $N \in \mathbb{N}$ such that $\langle X, \widetilde{\mu}\rangle - \phi'^*(\widetilde{\mu}) < 0$. Transforming $\widetilde{\mu}$ back to $\widetilde{a} \in \mathcal{A}_1^1$ via the Radon-Nikodym density $\Delta\widetilde{a}_t := \frac{\partial\widetilde{\mu}(.\ \cap \Omega\times\{t\})}{\partial\mathbb{P}}$ restricted to $\Omega \times \{t\}$, we see that $\widetilde{a} \in \mathcal{A}_1^{1,loc}$ and

$$\langle X, \widetilde{a}\rangle - \phi^*(\widetilde{a}) < 0.$$

This shows that $\mathcal{C}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^{1,loc})$-closed.

$(ii) \Rightarrow (iii)$: The $\sigma\left(\mathcal{R}^\infty, \mathcal{A}^{1,\mathrm{loc}}\right)$-upper semicontinuity follows directly from (ii) and the translation invariance.

$(iii) \Rightarrow (iv)$: This follows from the Fenchel-Moreau theorem.

$(iv) \Rightarrow (v)$: Fix $\varepsilon > 0$ and let (X^n) be a sequence in $\mathcal{R}^\infty$ and $X \in \mathcal{R}^\infty$ such that X_t^n is bounded and $X_t^n \to X_t$ $\mathbb{P}$-almost surely for all $t \in \mathbb{N}$. There is $a^* \in \mathcal{A}_1^{1,\mathrm{loc}}$ such that

$$\begin{aligned}
\phi(X) + \varepsilon &\geq \langle X, a^*\rangle - \phi^*(a^*) \\
&= \lim_{n\to\infty} \left(\langle X^n, a^*\rangle - \phi^*(a^*)\right) \\
&\geq \limsup_{n\to\infty} \inf_{a\in\mathcal{A}_1^{1,\mathrm{loc}}} \{\langle X^n, a\rangle - \phi^*(a)\} = \limsup_{n\to\infty} \phi(X^n).
\end{aligned}$$

$(v) \Rightarrow (i)$: Let $X \in \mathcal{R}^\infty$ such that $X\mathbb{1}_{(0,t)} + N(t)\mathbb{1}_{[t,\infty)} \in \mathcal{C}$ for all $t \in \mathbb{N}$ and some large $N(t) \in \mathbb{N}$. The sequence $(X^n)_{n\in\mathbb{N}}$ defined as

$$X^n := X\mathbb{1}_{(0,n)} + N(n)\mathbb{1}_{[n,\infty)}$$

satisfies $\phi(X^n) \geq 0$ for all $n \in \mathbb{N}$. Moreover X_t^n is bounded and $X_t^n \to X_t$ $\mathbb{P}$-almost surely for all $t \in \mathbb{N}$. Hence

$$\phi(X) \geq \limsup_{n\to\infty} \phi(X^n) \geq 0,$$

showing that $X \in \mathcal{C}$. This is equivalent to (A1). ☐

Remark 3.1. Let ϕ be a risk assessment which is locally continuous from below and has the robust representation

$$\phi(X) = \inf_{a\in\mathscr{A}} \{\langle X, a\rangle - \phi^*(a)\} \quad \text{for all } X.$$

The arguments in step 3 of the implication $(i) \Rightarrow (ii)$ above shows that for any $\gamma > 0$ the set $\mathscr{A}^\gamma := \{a \in \mathscr{A} : \phi^*(a) \geq -\gamma\}$ is locally uniformly integrable in the following sense:
For all $T \in \mathbb{N}$ and all $B^\kappa = (B_1^\kappa, \ldots)$ with $B_t^\kappa \nearrow \Omega$ for all $t \leq T$ and $B_t^\kappa = \Omega$ for all $t > T$ we have

$$\lim_{\kappa\to\infty} \inf_{a\in\mathscr{A}^\gamma} \langle \mathbb{1}_{B^\kappa}, a\rangle = 1$$

where $\mathbb{1}_{B^\kappa} = \big(\mathbb{1}_{B_1^\kappa}, \ldots\big)$. Indeed, for every $\varepsilon > 0$ and $M \geq \gamma/\varepsilon$ we get

$$\begin{aligned}\lim_{\kappa\to\infty} \inf_{a\in\mathscr{A}^\gamma} \langle \mathbb{1}_{B^\kappa}, a\rangle &\geq \frac{1}{M} \lim_{\kappa\to\infty} \inf_{a\in\mathcal{A}^\gamma} (\langle M\cdot \mathbb{1}_{B^\kappa}, a\rangle - \phi^*(a)) - \varepsilon \\ &\geq \frac{1}{M} \lim_{\kappa\to\infty} \phi\left(M\cdot \mathbb{1}_{B^\kappa}\right) - \varepsilon = 1-\varepsilon.\end{aligned}$$

A first example of a risk assessment which is asymptotically stable and locally continuous from below is the following

Example 3.1. Let $\mathcal{P}$ be a uniformly integrable set of absolutely continuous probabilities and $\mathbb{T}$ a subset of $\mathbb{N}$. Then

$$\phi(X) := \inf_{t\in\mathbb{T}} \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[X_t\right]$$

is a coherent, asymptotically stable, and locally continuous from below risk assessment.

Let $\phi : \mathcal{R}^\infty \to \mathbb{R}$ be a risk assessment which is locally continuous from below and satisfies (A1) ($\Leftrightarrow$ (A2)). Then it holds that

$$\phi(X) = \limsup_{t\to\infty} \phi(X\mathbb{1}_{(0,t)} + X_t\mathbb{1}_{[t,\infty)}) \tag{3.11}$$

However, the following example shows that there exist asymptotically stable risk assessments which are locally continuous from below, but $\phi(X) > \liminf_{t\to\infty} \phi(X\mathbb{1}_{(0,t)} + X_t\mathbb{1}_{[t,\infty)})$ for some $X \in \mathcal{R}^\infty$, i.e. the limes $\lim_{t\to\infty} \phi(X\mathbb{1}_{(0,t)} + X_t\mathbb{1}_{[t,\infty)})$ does not exist as the following example shows:

Example 3.2. Suppose $\Omega = \{\omega\}$ and $\phi(X) := \inf_{a\in\mathcal{Q}} \langle X, a\rangle$ where

$$\mathcal{Q} := \left\{a \in \mathcal{A}_1^{1,loc} : \Delta a_t = \Delta a_{t+1} = \frac{1}{2} \text{ for some } t \in \mathbb{N}\right\}.$$

According to Theorem 3.2, ϕ is locally continuous from below and satisfies $(A1) \Leftrightarrow (A2)$. However, for $X = (1, -1, 1, -1, 1, \ldots)$ one has $\phi(X) = 0$, while $\liminf_{t\to\infty} \phi(X\mathbb{1}_{(0,t)} + X_t\mathbb{1}_{[t,\infty)}) = -1$. In particular, the limit $\lim_{t\to\infty} \phi(X\mathbb{1}_{(0,t)} + X_t\mathbb{1}_{[t,\infty)})$ does not exist.

4. Dynamic Risk Assessments

In this section, we study families of risk assessments $(\phi_s)_{s\in\mathbb{N}_0}$ which are time-consistent, that is

$$\phi_s(X) = \phi_s\left(X\mathbb{1}_{(s,t]} + \phi_t(X)1_{(t,\infty)}\right), \quad \text{for all } s,t\in\mathbb{N} \text{ with } s<t. \tag{4.1}$$

We here work with the notion of time-consistency as in [10] which is slightly different to the respective notion in [5] and [1]. Since both concepts of time-consistency are formally very similar, the results of this section can directly be adapted to the context of [5] and [1].

Time-consistent risk assessments lead naturally the the notion of a family of generators $G_s : L^\infty_{s+1}\times L^\infty_{s+1}\to L^\infty_s$ by defining

$$G_s(Z^1,Z^2) := \phi_s\left(Z^1\mathbb{1}_{\{s+1\}} + Z^2\mathbb{1}_{(s+1,\infty)}\right). \tag{4.2}$$

This gives

$$\phi_s(X) = G_s\left(X_{s+1},\phi_{s+1}(X)\right). \tag{4.3}$$

Here the goal is to give conditions of a family $(G_s)_{s\in\mathbb{N}_0}$ of generators which leads to asymptotically stable and locally continuous from below risk assessments. We start with the following properties of generators:

(G0) $G_s(0,0)=0$,

(G1) $G_s(X+m,Y+m) = G_s(X,Y)+m$ for all $m\in L^\infty_s$,

(G2) $G_s(X^1,Y^1)\geq G_s(X^2,Y^2)$ whenever $X^1\geq X^2$ and $Y^1\geq Y^2$,

(G3) $G_s(X,Y) = \lim_{n\to\infty} G_s(X^n,Y^n)$ for every decreasing sequence (X^n,Y^n) which converges to some (X,Y) $\mathbb{P}$-almost surely,

(G3') $G_s(X,Y) = \lim_{n\to\infty} G_s(X^n,Y^n)$ for every increasing sequence (X^n,Y^n) which converges to some (X,Y) $\mathbb{P}$-almost surely,

(G4) $G_s(\lambda X^1+(1-\lambda)X^2,\lambda Y^1+(1-\lambda)Y^2)\geq \lambda G_s(X^1,Y^1)+(1-\lambda)G_s(X^2,Y^2)$ for all $\lambda\in L^\infty_s$ with $0\leq\lambda\leq 1$.

Under the concavity assumption (G4) the condition (G3') implies (G3).

For $X\in\mathcal{R}^\infty$, $s\in\mathbb{N}_0$ and $N\geq\|X\|_{\mathcal{R}^\infty}$ the sequence $\{G_s(X_{s+1},\cdot)\circ\cdots\circ G_{t-1}(X_t,N)\}_{t\geq s+1}$ is decreasing in t and we define

$$\phi^N_s(X) := \inf_{t\geq s+1} G_s(X_{s+1},\cdot)\circ\cdots\circ G_{t-1}(X_t,N).$$

Theorem 4.1. *Suppose that the generators $(G_s)_{s\in\mathbb{N}_0}$ satisfy (G0)–(G3) and*

$$\lim_{n\to\infty} G_s(0,\cdot)\circ\cdots\circ G_{s+n}(0,m)=0 \quad \text{for all } m\geq 0. \tag{4.4}$$

Then, for every $s\in\mathbb{N}_0$, it holds

$$\phi_s(X):=\phi_s^N(X)=\phi_s^M(X) \quad \text{for all } M,N\geq \|X\|_{\mathcal{R}^\infty},$$

and the family $(\phi_t)_{t\in\mathbb{N}_0}$ is time-consistent in the sense of (4.1)*. Further, ϕ_0 satisfies the properties (i)–(iii) of Definition 2.1 and $\phi_0(X)=\lim_{n\to\infty}\phi_0(X^n)$ for every decreasing sequence (X^n) in $\mathcal{R}^\infty$ such that $X^n_t\to X_t$ $\mathbb{P}$-almost surely for all $t\in\mathbb{N}$ for some $X\in\mathcal{R}^\infty$.*
Under the additional assumption (G4), ϕ_0 is a concave risk assessment.
If the generators satisfy (G3') instead of (G3) then ϕ_0 is locally continuous from below.

Remark 4.1. The condition (4.4) is for instance satisfied if for every $\varepsilon>0$ there exists $\beta(\varepsilon)$ with $0\leq\beta(\varepsilon)<1$ such that $G_s(0,m)\leq\beta(\varepsilon)m$ for all $m\geq\varepsilon$ and $G_s(0,m)\leq\varepsilon$ whenever $m<\varepsilon$ for eventually all s.

Proof. We first show that the definition of ϕ_s^N does not depend on $N\geq\|X\|_{\mathcal{R}^\infty}$. To that end, we fix $X\in\mathcal{R}^\infty$ and $M\geq N\geq\|X\|_{\mathcal{R}^\infty}$. For every $\varepsilon>0$ there exists $t\in\mathbb{N}$ such that

$$\varepsilon+\phi_0^N(X)\geq G_0(X_1,\cdot)\circ\cdots\circ G_{t-1}(X_t,N).$$

In view of (G1) it holds

$$G_t(N,\cdot)\circ\cdots\circ G_{t'-1}(N,M)=N+G_t(0,\cdot)\circ\cdots\circ G_{t'-1}(0,M-N)$$

for all $t'\geq t+1$. Thus, by condition (4.4), (G1) and (G2) there exists $t'\geq t$ such that

$$\begin{aligned}
\phi_0^N(X)+2\varepsilon &\geq G_0(X_1,\cdot)\circ\cdots\circ G_{t-1}(X_t,N)+\varepsilon\\
&\geq G_0(X_1,\cdot)\circ\cdots\circ G_{t-1}(X_t,\cdot)\circ G_t(N,\cdot)\circ\cdots\circ G_{t'-1}(N,M)\\
&\geq G_0(X_1,\cdot)\circ\cdots\circ G_{t-1}(X_t,\cdot)\circ G_t(X_{t+1},\cdot)\circ\cdots\circ G_{t'-1}(X_{t'},M)\\
&\geq \phi_0^M(X).
\end{aligned}$$

Since $\phi_0^N(X)\leq\phi_0^M(X)$ by (G2), we get $\phi_0^N(X)=\phi_0^M(X)$, which shows that ϕ_0 is well-defined. The argumentation for ϕ_s works analogously, however t and t' have to be chosen $\mathcal{F}_s$-measurable with values in $\mathbb{N}$.

As for the time-consistency (4.1), the conditions (G2) and (G3) imply

$$\begin{aligned}
&\phi_s\left(X\mathbb{1}_{(s,t]}+\phi_t(X)\mathbb{1}_{(t,\infty)}\right)\\
=&G_s(X_{s+1},\cdot)\circ\cdots\circ G_{t-1}(X_t,\phi_t(X))\\
=&G_s(X_{s+1},\cdot)\circ\cdots\circ G_{t-1}(X_t,\cdot)\circ\left(\inf_{t'\geq t+1}G_t(X_{t+1},\cdot)\circ\cdots\circ G_{t'-1}(X_{t'},N)\right)\\
=&\inf_{t'\geq t+1}G_s(X_{s+1},\cdot)\circ\cdots\circ G_{t-1}(X_t,\cdot)\circ G_t(X_{t+1},\cdot)\circ\cdots\circ G_{t'-1}(X_{t'},N)\\
=&\phi_s(X),
\end{aligned}$$

for all $N \geq \|X\|_{\mathcal{R}^\infty}$.

To show that ϕ_0 is continuous from above, let (X^n) be a decreasing sequence in $\mathcal{R}^\infty$ such that $N := \sup_{n\in\mathbb{N}} \|X^n\|_{\mathcal{R}^\infty} < \infty$ and $X_t^n \to X_t$ P-almost surely for all $t \in \mathbb{N}$ for some $X \in \mathcal{R}^\infty$. For every $\varepsilon > 0$ there exists a time horizon $t \in \mathbb{N}$ such that

$$\begin{aligned}\phi_0(X) + \varepsilon &\geq G_0(X_1,\cdot)\circ\cdots\circ G_{t-1}(X_t, N))\\ &= \lim_{n\to\infty} G_0(X_1^n,\cdot)\circ\cdots\circ G_{t-1}(X_t^n, N)\\ &\geq \lim_{n\to\infty} \phi_0(X^n).\end{aligned}$$

On the other hand, $\phi_0(X^n) \geq \phi_0(X)$ for all $n \in \mathbb{N}$, so that $\lim_{n\to\infty} \phi_0(X^n) = \phi_0(X)$.

In case that the generators satisfy (G3'), for all $T \in \mathbb{N}$ we have

$$\begin{aligned}\phi_0(X\mathbb{1}_{(0,T)} + X_T\mathbb{1}_{[T,\infty)}) &= G_0(X_1,\cdot)\circ\cdots\circ G_{T-1}(X_T, X_T))\\ &= \lim_{n\to\infty} G_0(X_1^n,\cdot)\circ\cdots\circ G_{T-1}(X_T^n, X_T^n)\\ &= \lim_{n\to\infty} \phi_0(X^n\mathbb{1}_{(0,T)} + X_T^n\mathbb{1}_{[T,\infty)})\end{aligned}$$

for every sequence (X^n) in $\mathcal{R}_T^\infty$ which increases to some $X \in \mathcal{R}_T^\infty$, which shows that ϕ_0 is locally continuous from below. □

Proposition 4.2. *Suppose that the generators $(G_s)_{s\in\mathbb{N}_0}$ satisfy (G0)–(G3') and condition* (4.4). *If in addition*

$$\limsup_{t\to\infty} \sup_{N\in\mathbb{N}} G_t(0, N) < \infty, \tag{4.5}$$

then ϕ_0 is asymptotically stable.

Proof. For every $X \in \mathcal{R}^\infty$ and $N \in \mathbb{N}$ one has

$$\begin{aligned}\phi_0\left(X\mathbb{1}_{(0,t]} + N\mathbb{1}_{(t,\infty)}\right) &= \phi_0\left(X\mathbb{1}_{(0,t)} + \phi_{t-1}\left(X\mathbb{1}_{\{t\}} + N\mathbb{1}_{(t,\infty)}\right)\mathbb{1}_{[t,\infty)}\right)\\ &\leq \phi_0\left(X\mathbb{1}_{(0,t)} + G_{t-1}(\|X\|_{\mathcal{R}^\infty}, \|X\|_{\mathcal{R}^\infty} + N)\mathbb{1}_{[t,\infty)}\right)\\ &\leq \phi_0\left(X\mathbb{1}_{(0,t)} + [\|X\|_{\mathcal{R}^\infty} + G_{t-1}(0,N)]\,\mathbb{1}_{[t,\infty)}\right)\end{aligned}$$

for all $t \in \mathbb{N}$. Hence, by (4.5) there exist $t_0 \in \mathbb{N}$ and a constant $C \in \mathbb{N}$ such that

$$\sup_{N\in\mathbb{N}}\left(\phi_0\left(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}\right) - \phi_0(X)\right) \leq \phi_0\left(X\mathbb{1}_{(0,t)} + (\|X\|_{\mathcal{R}^\infty} + C)\,\mathbb{1}_{[t,\infty)}\right) - \phi_0(X)$$

for all $t \geq t_0$. Since ϕ_0 is continuous from above, the right hand side tends to zero as t goes to infinity. Hence

$$\lim_{t\to\infty}\sup_{N\in\mathbb{N}}\left(\phi_0\left(X\mathbb{1}_{(0,t)} + N\mathbb{1}_{[t,\infty)}\right) - \phi_0(X)\right) = 0,$$

showing that ϕ_0 satisfies (A2). □

5. Examples

In this section, we construct examples of generators which satisfy the conditions (G0)–(G3'), (4.4) and (4.5). To that end, we consider generators of the form

$$G_s(X,Y) := \psi_s\left(X + h_s(Y - X)\right), \quad s \in \mathbb{N}_0, \tag{5.1}$$

where $\psi_s : L^\infty_{s+1} \to L^\infty_s$ such that

(p0) $\psi_s(0) = 0$,

(p1) $\psi_s(Z + m) = \psi_s(Z) + m$ for all $m \in L^\infty_s$,

(p2) $\psi_s(Z^1) \geq \psi_s(Z^2)$ whenever $Z^1 \geq Z^2$,

(p3) $\psi_s(Z^n) \to \psi_s(Z)$ for every sequence (Z^n) which increases to Z,

(p4) $\psi_s(\lambda Z^1 + (1-\lambda)Z^2) \geq \lambda\psi_s(Z^1) + (1-\lambda)\psi_s(Z^2)$ for all $\lambda \in L^\infty_s$ with $0 \leq \lambda \leq 1$,

and the function $h_s : \mathbb{R} \to \mathbb{R}$ satisfies

(h0) $h_s(0) = 0$,

(h1) $h_s(z + m) \leq h_s(x) + m$ for all $z \in \mathbb{R}$ and $m \geq 0$,

(h2) $h_s(z^1) \geq h_s(z^2)$ whenever $z^1 \geq z^2$,

(h3) h_s is continuous,

(h4) h_s is concave.

A straightforward application of Theorem 4.1, Remark 4.1 and Proposition 4.2 is then:

Proposition 5.1. *Let $(G_s)_{s\in\mathbb{N}_0}$ be a sequence of generators of the form* (5.1) *which satisfy (p0)–(p4), (h0)–(h4), for every $\varepsilon > 0$ there exists $0 \leq \beta(\varepsilon) < 1$ such that $h_s(m) \leq \beta(\varepsilon)m$ for all $m \geq \varepsilon$ and $h(m) \leq \varepsilon$ whenever $m < \varepsilon$, and*

$$\limsup_{s\to\infty} \sup_{z\in\mathbb{N}} h_s(z) < \infty.$$

Then the generators G_s satisfy (G0)–(G2), (G3'), (G4) and the corresponding concave risk assessment ϕ_0 is locally continuous from below and aymptotically stable.

Proof. Clearly, such generators G_s satisfy (G0) and (G1). As for the monotonicity (G2), for $X^1 \geq X^2$ and $Y^1 \geq Y^2$ we have $X^2 + h_s(Y^2 - X^2) \leq X^1 + h_s(Y^2 - X^1)$ by (h1) so that

$$\begin{aligned} G_s(X^1,Y^1) &= \psi_s(X^1 + h_s(Y^1 - X^1)) \geq \psi_s(X^1 + h_s(Y^2 - X^1)) \\ &\geq \psi_s(X^2 + h_s(Y^2 - X^2)) = G_s(X^2,Y^2). \end{aligned}$$

To show (G3') let (X^n, Y^n) be a sequence which increases to (X, Y). Then $X^n + h_s(Y^n - X^n)$ increases to $X + h_s(Y - X)$ by (h1)–(h3) so that $G_s(X^n, Y^n)$ increases to $G_s(X, Y)$. By (p2) and (h4), it holds that

$$\begin{aligned}
&\psi_s\left(\lambda X^1 + (1-\lambda)X^2 + h_s(\lambda(Y^1 - X^1) + (1-\lambda)(Y^2 - X^2))\right) \\
&\geq \psi_s\left(\lambda X^1 + (1-\lambda)X^2 + \lambda h_s(Y^1 - X^1) + (1-\lambda)h_s(Y^2 - X^2)\right) \\
&\geq \lambda\psi_s(X^1 + h_s\left(Y^1 - X^1\right)) + (1-\lambda)\psi_s(X^2 + h_s\left(Y^2 - X^2\right)),
\end{aligned}$$

which shows (G4).

Finally, that ϕ_0 is an asymptotically stable risk assessment follows from Theorem 4.1 and Proposition 4.2 since

$$G_s(0, m) = \psi_s(h_s(m)) = h_s(m)$$

implies (4.4) by Remark 4.1 and (4.5). □

Notice that Proposition 5.1 allows to construct time-consistent risk assessments which are asymptotically stable. For instance, the generators

$$G_s(X, Y) = \psi_s\left(X + h_s(Y - X)\right)$$

can be defined through the negative of a conditional risk measure ψ_s such as the entropic utility function

$$\psi_s(Z) = \frac{1}{\gamma}\log\left(\mathbb{E}\left[\exp(-\gamma Z) \mid \mathcal{F}_s\right]\right)$$

with risk aversion parameter γ, and discounting functions such as

- $h_s(z) := -z^-$
- $h_s(z) = 1 - \exp(-z^+) - z^-$,

where $z^+ := \max(x, 0)$ and $z^- := \max(-z, 0)$, and which both satisfy the assumptions of Proposition 5.1. Alternatively, ψ_s could be chosen as the negative of the conditional Value at Risk for which (p4) does not hold, or the conditional Average Value at Risk.

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