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Abstract

This paper is based on a general method for multiperiod prudential supervision of companies submitted to hedgeable and non-hedgeable risks. Having treated the case of insurance in an earlier paper, we now consider a quantitative approach to supervision of commercial banks.

The various elements under supervision are the bank's current amount of tradeable assets, the deposit amount, and four flow processes: future trading risk exposures, deposit flows, flows of loan repayments and of deposit remunerations. The approach uses a multiperiod risk assessment supposed not to allow supervisory arbitrage. Coherent and non-coherent examples of such risk assessments are given.

The risk assessment is applied to the risk bearing capital process composed out of the amounts of assets and deposits, and the four flow processes mentioned above. We give a general definition of a *supervisory margin* which uses the risk assessment under the assumption of optimal trading risk exposures. The transfer principle together with a cost-of-capital ratio gives quantitative definitions of the *risk margin* and of the *non-hedgeable equity capital requirement*. The *hedgeable equity capital requirement* measures the inadequacy of the bank's portfolio of tradeable assets with respect to the optimal trading risk exposures.

The hierarchy of different interferences of a supervisor is related to these quantities. Finally, a simple allocation principle for margins and the equity capital requirements is derived.

Key words and phrases: *equity capital requirements, hierarchy of supervisor's interferences, multiperiod risk assessment, optimal trading risk exposures, supervisory margin.*

AMS - Classification: *90B50, 91Gxx.*

JEL - Classification: *G18, G21, G32.*

1. Introduction

In this paper we study models for the supervision of banks with concessions for deposits. In a nutshell, such banks collect money from depositors and distribute it to borrowers at an interest rate higher than the remuneration rate on deposits. The group of borrowers generate by their potential defaults some external in general non-tradeable risks. The market consistent value of the loans is one of the prospective parts of the bank's active balance sheet. On the other hand we find the group of depositors. The amount of their deposits is on the passive side of the balance sheet and it is a task of banking supervision to protect the depositors from the risk created by borrowers while accepting a great liberty in the randomness of timings and amounts of deposits. A model in which deposit remuneration is higher than that of the eligible asset could also be used to study cases where the bank has exogenous financial commitments as in the case of "bank-insurance".

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We present a formal model of banking supervision in a multiperiodic stochastic framework. Next to the processes of traded securities prices, the extra stochastic elements are the cash flows of loan repayments, deposits and commitments. They are all grouped together in the bank's business plan.

As in Eisele and Artzner (2011), we use risk assessments and differentiate the accounting between the active and the passive components of the *supervisory balance sheet*. The argument of the risk assessment functionals is the *risk bearing capital* associated to the business plan. It is composed out of four quantities: The current amount of tradeable assets, the best estimate valuation of the loan repayment flows via a market risk neutral probability measure, the current amount of deposits and the best estimate valuation of commitments. Here the amounts are *retrospective* quantities while the valuations are *prospective*. In our study the flows of loans, deposits and commitments are exogenously given, however the flows stemming from the company's tradeable asset portfolio are endogenous, i.e. they serve as control variables for the manager.

The interaction of market tools and supervisory assessment creates opportunities for the good and for the bad. A business position deemed critical in the eyes of the supervisor, may sometimes be improved at zero cost just by a rebalancing of assets via traded instruments, but there should not exist a zero cost portfolio allowing this on a systematic basis. This is examined in Definition 3.4 and Proposition 3.1 under the headings of absence of a *supervisory arbitrage*. The non-arbitrage property allows also to introduce the *market consistent hull of a risk assessment* which turns out to be essential for defining supervisory margins.

Solvability (not acceptability) of a business plan is defined by positivity of the market consistent hull of the risk assessment applied to the risk bearing capital. At this point the supervisory considerations are done under the hypothesis of an *optimal process of trading risk exposures* (also called optimal replicating portfolio). The positive difference between the initial value of the risk bearing capital and its prudent evaluation (as a process) via the market consistent hull of the risk assessment is the *supervisory margin*.

Acceptability means the positivity of the original risk assessment. This means that if the bank's manager insists on a trading portfolio deviating from the optimal trading risk exposures, he has to guaranty an additional amount of equity capital, the *hedgeable equity capital requirement*, in order to become acceptable. The hedgeable equity capital requirement is just the difference between the assessments of the process of risk bearing capital first under optimal trading risk exposures and second under those given in the business plan. On the other hand, by the cash invariance property the result of the original risk assessment is also the company's *free capital* which — if positive — the bank can dispense with and still stay acceptable.

In absence of solvability the protection of the depositors may in principle still be granted by a transfer of loans and commitments to a new bank. The new bank, called reference bank, is supposed to have a zero free capital and optimal trading exposures. The regulated transfer price differs from the market consistent best estimate of the loans and commitments. The difference is called the *risk margin of loans and commitments*. It is external capital and serves as expected gain for the new bank (neglecting limited liability) within the supervisory horizon. But the new bank has also to invest equity capital (and is willing to do so) in order to satisfy acceptability and to gain the risk margin. This investment is the *non-hedgeable equity capital requirement* which is hard equity capital to be collected from the shareholders, since it cannot be replaced by deposits or obligations.

This thought-of transaction is *regulated* since supervision decides upon the ratio (also known as *cost-of-capital ratio*) between the risk margin and the non-hedgeable equity capital requirement. Risk margin and non-hedgeable equity capital requirement add up to the supervisory margin, which thus turns out to be a mixture of equity and external capital.

In insurance the aim of supervision is the protection of the policy-holders whose contracts generate the underwriting risks: The group of people who should be protected and the one who create the external, in general non-tradeable risks are identical. In banking the extra group of agents, the borrowers, create a risk for the creditors. Therefore banking supervision requires a more elaborate use of risk assessments although the steps acceptability, solvability, and transferability according to the amount of available equity capital are present in both theories. In fact recently financial industry experiences developments towards similar supervision systems as Basel III and Solvency II, and a tendency to common accounting standards. Another significant feature is the creation of supervisory institutions dealing with both banks and insurance companies like OSFI, Canada in 1987, BAFIN, Germany in 2002, FMA, Austria in 2002, FSA, UK in 2005, FINMA, Switzerland in 2009, and ACP, France in 2010. By Eisele and Artzner (2011) and the present paper, we hope to contribute to this convergence.

2. A financial market

The time space is given by $\mathbb{T} := \{0, 1, \ldots, T\}$ for some $T \in \mathbb{N}$. Random variables and stochastic processes are defined on a finite filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ with Ω the support of $\mathbb{P}$.
Let L_t be the set of $\mathcal{F}_t$-measurable random variables and $L = \prod_{t\in\mathbb{T}} L_t$ the set of adapted stochastic processes in the time space $\mathbb{T}$. L is a closed subspace of $\mathbb{R}^{\mathbb{T}\times\Omega}$. By L_t^+ (resp. L^+) we denote the space of all non-negative random variable $x_t \geq 0$ in L_t (resp. of non-negative processes $X \geq 0$ in L). Similarly, $\overline{L}_t^+$ (resp. $\overline{L}^+$) is the set of $\mathcal{F}_t$-measurable random variables (resp. adapted stochastic processes) with values in $[0, +\infty]$. The random variables x_t in L_t represents financial amounts x_t of date t money.

Let $0 < t \leq T$. For a probability measure $\mathbb{Q} \ll \mathbb{P}$, absolutely continuous w.r.t. $\mathbb{P}$, we define the $\mathcal{F}_{t-1}$-conditional Radon Nikodym derivative of $\mathbb{Q}$ restricted to $\mathcal{F}_t$ by

$$\partial\mathbb{Q}_t := \frac{\mathbb{E}\left[\partial\mathbb{Q}/\partial\mathbb{P}\middle|\mathcal{F}_t\right]}{\mathbb{E}\left[\partial\mathbb{Q}/\partial\mathbb{P}\middle|\mathcal{F}_{t-1}\right]} \tag{2.1}$$

with the convention $0/0 = 0$.

The following obvious short-hand writing will be useful: We write $\mathbb{I}_{\leq t}$ instead of $\mathbb{I}_{\{u, u\leq t\}}$, similarly for expressions like for $\mathbb{I}_{=t}$ or $\mathbb{I}_{<t}$. Moreover, we combine this with stochastic processes and random variables: for example, if $X \in L$ and $y_{t+1} \in L_{t+1}$ then $X\mathbb{I}_{\leq t} + y_{t+1}\mathbb{I}_{>t}$ is the process with values X_u at times $u \leq t$ and the constant value y_{t+1} at all times $u > t$.

A financial market is given by a family of asset processes $\left(S^i\right)_{i=0,\ldots,d}$ in L whose values of date t money are the random variables $S_t^i \in L_t$. Their current market prices are S_0^i.

We take S^0 as *numéraire*, meaning simply that $S_0^0 = 1$ and $S_t^0(\omega) > 0$ for all $\omega \in \Omega$ and $t \in \mathbb{T}$.

We suppose that the set of trading-risk neutral probabilities is not empty: $\mathcal{M} \neq \emptyset$ where

$$\mathcal{M} := \left\{\mathbb{Q}\middle| \mathbb{E}_{\mathbb{Q}}\left[S_{t+1}^i \frac{S_t^0}{S_{t+1}^0}\middle|\mathcal{F}_t\right] = S_t^i \; i = 1, \ldots, d,\; 0 \leq t < T\right\}. \tag{2.2}$$

Note that $\mathcal{M}$ is a m-stable set of probabilities in the sense of Delbaen (2006) Definition 1.2. For the period $(t, t+1]$ with $0 < t \leq T$ the set of conditional trading-risk neutral densities is

$$\partial\mathcal{M}_t := \left\{\partial\mathbb{Q}_t \in L_t^+\middle| \mathbb{Q} \in \mathcal{M}\right\}. \tag{2.3}$$

The set of flows of zero-cost portfolios in the period $[t-1, t]$, $t > 0$ is

$$N_t := \left\{a_t = \sum_{i=0}^d S_t^i \cdot \xi_{t-1}^i \middle| \sum_{i=0}^d S_{t-1}^i \cdot \xi_{t-1}^i = 0,\; \xi_{t-1}^i \in L_{t-1}\right\}. \tag{2.4}$$

We call $a_t \in N_t$ a **trading-risk exposure** at t and

$$N := \{0\} \times \prod_{t=1}^{T} N_t \tag{2.5}$$

is the set of cash-flows processes of trading-risk exposures.

3. Eligible asset and risk assessment

An *eligible asset* $\rho \in L$ is a financial asset (or the value of a portfolio) which from the supervisor's point of view provides a benchmark of a (relatively) risk-free investment. In particular, ρ_t is strictly positive and we normalize the eligible asset to $\rho_0 = 1$. There is no mathematical restriction in treating numéraire and eligible asset as equal: $\rho = S^0$ (see Artzner and Eisele (2010)).

For $s, t \in \mathbb{T}$, we set $\rho_{s,t} := \rho_t / \rho_s$. The **saving process** $\rho(s)$ in the eligible asset ρ, starting at $s \in \mathbb{T}$, is given by

$$\rho(s) := \frac{1}{\rho_s}\, \rho\, \mathbb{1}_{\geq s}. \tag{3.1}$$

Obviously, $\rho(s)_t = \rho_{s,t}$ for $s \leq t$.

Now we can introduce for a process X its increment process ∂X by

$$\partial X_t := X_t - X_{t-1} \cdot \rho_{t-1,t} \tag{3.2}$$

for $t \in \mathbb{T}$, with the convention $\partial X_0 = 0$. The following relation between the value process $X \in L$ and the increment process ∂X holds:

$$X \mathbb{1}_{\geq t} = X_t \cdot \rho(t) + \sum_{u=t+1}^{T} \partial X_u \cdot \rho(u). \tag{3.3}$$

Indeed, at date $v \geq t$ the process on the right-hand side of (3.3) becomes a telescopic sum $X_t \cdot \rho_{t,v} + \sum_{u=t+1}^{v} (X_u - X_{u-1} \cdot \rho_{u-1,u}) \cdot \rho_{u,v} = X_v$. Since we keep the eligible asset ρ fixed we suppress the reference to ρ in the following definitions.

Let $\mathbb{Q}$ be a probability measure on $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}})$. A process $X \in L$ is called a **discounted $\mathbb{Q}$-martingale** (resp. a **discounted $\mathbb{Q}$-supermartingale**) if and only if for all $t < T$

$$\mathbb{E}_{\mathbb{Q}}\big[\partial X_{t+1} \cdot \rho_{t+1,t} \big| \mathcal{F}_t\big] = 0 \quad \text{(resp.} \quad \leq 0). \tag{3.4}$$

By $S_{\mathbb{Q}}$ we denote the space of all discounted $\mathbb{Q}$-supermartingales. $S_{\mathbb{Q}}$ is a closed cone in L.

Definition 3.1.
Given an eligible asset ρ, a **risk assessment** $\Psi = (\Psi_t)_{t \in T}$ is a family of functionals $\Psi_t : L \to L_t$ with the following properties: For all $X, X^{(n)}, Y \in L$, $s \leq t \in \mathbb{T}$, $y_t \in L_t$, and $A \in \mathcal{F}_t$

(i) Final assessment: $\Psi_T = 0$,
(ii) Localization: $\Psi_t(X \cdot \mathbb{1}_{>t} \cdot \mathbb{1}_A) = \Psi_t(X \cdot \mathbb{1}_{>t}) \cdot \mathbb{1}_A$,
(iii) Monotonicity on future values: if $X \mathbb{1}_{>t} \leq Y \mathbb{1}_{>t}$ then $\Psi_t(X) \leq \Psi_t(Y)$,

(iv) Cash invariance with respect to ρ: $\Psi_t(X + y_t \cdot \rho(t)) = \Psi_t(X) + y_t$,
(v) Lower semi-continuity: If $X^{(n)} \nearrow X$ in L for $n \to \infty$, then $\Psi_t(X^{(n)}) \nearrow \Psi_t(X)$,
(vi) Time consistency: $\Psi_s(X) = \Psi_s(X \mathbb{1}_{\leq t} + \Psi_t(X) \cdot \rho(t)\ \mathbb{1}_{>t})$.

Remark 3.1.
(i) Note that Definition 3.1 does not contain any concavity or coherence condition.
(ii) The mathematical formulations of cash invariance and time consistency imply that the processes X represent cumulative amounts of money.
(iii) Definition 3.1 shows that risk assessment Ψ and eligible asset ρ form a pair (Ψ, ρ). This is in analogy with the definition of risk-neutral probabilities which should always be defined with respect to a numéraire S^0. Most textbooks and articles however assume $S^0 \equiv 1$ such that a risk-neutral probability is seemingly independent of a choice of numéraire. But this is clearly an artificial trompe-l'œil.
For simplicity however, we continue to write Ψ even if a notation like Ψ_ρ would be more appropriate. For example, a change of the numéraire has to go hand in hand with a change of the risk assessment if the acceptance set is kept invariant (see also the discussion in Artzner et al. (2009) Section 3).
By the way, the last consideration gives a hint why law-invariant risk assessments are in general not of great help in supervisory respects.

Though the concept of a "market consistent evaluation" is since long used in the financial industry, a precise definition has only be given in Cheridito et al. (2008).

Definition 3.2.
Let $S \subset L$ with $S + S_{\mathcal{M}} = S$. A family $\Phi = (\Phi_t)_{t\in\mathbb{T}}$ of functionals $\Phi_t : L \to L_t$ is called **market consistent** on S if for all $X \in S$ and $a \in N$

$$\Phi_t(X) = \Phi_t(X + \sum_{u>t} a_u \cdot \rho(u)). \tag{3.5}$$

The following *market consistent best estimates of final values* are important examples of market consistent functionals.

Definition 3.3 (Best estimate of final values)**.**
Let $\mathbb{Q} \in \mathcal{M}$. Then for $X \in L$

$$\begin{array}{rcll} \Psi_{\mathbb{Q},T}(X) & := & 0 & \text{and} \\ \Psi_{\mathbb{Q},t}(X) & := & \mathbb{E}_{\mathbb{Q}}\left[X_T \cdot \rho_{T,t} \middle| \mathcal{F}_t\right] & \text{for } t < T \end{array} \tag{3.6}$$

is called **best estimate** of X.

Definition 3.4.
Let $S \subseteq L$ with $S + S_{\mathcal{M}} = S$, and $\Phi = (\Phi_t)_{t\in\mathbb{T}}$ a family of functionals $\Phi_t : L \to L_t$.
(i) We say that Φ is **bounded by identity on** S if for all $X \in S$ and $0 \leq t < T$

$$\Phi_t(X) \leq X_t. \tag{3.7}$$

(ii) We say that Φ allows **arbitrage on** S if for some $X \in S$ and some t, $0 \leq t < T$,

$$\operatorname{ess.sup} \Phi_t(X + \sum_{u=t+1}^{T} a_u \cdot \rho(u)) = +\infty \tag{3.8}$$

where the ess . sup is taken over all $a \in N$.

The following result was shown in Eisele and Artzner (2011):

Proposition 3.1.
Let Φ be a family of functionals bounded on $S \subseteq L$ by a positive affine transformation of identity: there exist $b \in \mathbb{R}$, $c > 0$ with

$$\Phi_t(X) \leq b + c \cdot X_t \tag{3.9}$$

for all $X \in S$ and $t < T$. Then Φ does not allow arbitrage on S.

4. Market consistent hull, optimal trading risk exposures, and supervisory margin

The following important result was proved in Eisele and Artzner (2011).

Theorem 4.1.
Let $\mathbb{Q} \in \mathcal{M}$ and Ψ be a risk assessment bounded by identity on $S_{\mathbb{Q}}$. We define a family $\Psi^* = (\Psi_t^*)_{t\in\mathbb{T}}$ of functionals $\Psi_t^* : S_{\mathbb{Q}} \to L_t$ by $\Psi_T^* = 0$ and for $t < T$ by

$$\Psi_t^*(X) := \operatorname*{ess.sup}_{a\in N} \Psi_t \left(X + \sum_{u=t+1}^{T} a_u \cdot \rho(u) \right), \tag{4.1}$$

$X \in S_{\mathbb{Q}}$. Ψ^* is the **market consistent hull**[1] *of Ψ on $S_{\mathbb{Q}}$: i.e. Ψ^* is the least market consistent risk assessment on $S_{\mathbb{Q}}$ dominating Ψ .*

Definition 4.1.
A process $a^* \in N$ is called a **t-optimal process of trading risk exposures**[2] for $X \in \mathcal{S}_{\mathbb{Q}}$ if

$$\Psi_t^*(X) = \Psi_t \left(X + \sum_{u=t+1}^{T} a_u^* \cdot \rho(u) \right). \tag{4.2}$$

It has been shown in Artzner and Eisele (2010) that optimal trading risk exposures do not always exist. We need some additional assumptions in order to guarantee its existence (see Artzner and Eisele (2010) and Eisele and Artzner (2011) for details).

As an immediate consequence of Theorem 4.1 we get

Corollary 4.2.
If Ψ is a risk assessment bounded by identity on $S_{\mathbb{Q}}$ with $\mathbb{Q} \in \mathcal{M}$, then so is its market consistent hull Ψ^* on $S_{\mathbb{Q}}$.

The time consistency of the market consistent hull together with Lemma 3.2 in Eisele and Artzner (2011) allows for its recursive calculation:

Proposition 4.3.
Let Ψ as in Theorem 4.1. Then the market consistent hull satisfies for all $X \in S_{\mathbb{Q}}$:

(i) $\Psi_T^*(X) = 0$

[1] In Eisele and Artzner (2011) it was called *market consistent majorant*.
[2] In the context of Solvency II the expression of an *optimal replicating portfolio* for the liabilities is used.

(ii) for $t < T$

$$\Psi_t^*(X) = \operatorname*{ess.sup}_{a_{t+1}\in N_{t+1}} \Psi_t\Big(X\,\mathbb{1}_{\le t+1} + a_{t+1}\cdot\rho(t+1) + \Psi_{t+1}^*(X)\cdot\rho(t+1)\,\mathbb{1}_{>t+1})\Big). \tag{4.3}$$

Remark 4.1.
The comparison between (4.1) and (4.3) shows that the global definition of a t-optimal process of trading risk exposures given by (4.2), can be replaced equivalently by the local problem of optimality in (4.3). This fact corresponds to the well known *Bellman principle* (see Kall and Wallace (1994)). In our case the market consistent hull Ψ^* serves as Bellman function.

The following general definition will turn out to deliver the essential notion for the supervision of non-hedgeable risky positions.

Definition 4.2.
Let Ψ a risk assessment bounded by identity and $X \in \mathcal{S}_{\mathbb{Q}}$. The **supervisory margin** $SM(X) = (SM_t(X))_{t\in\mathbb{T}}$ of X is defined as

$$\begin{array}{rcllr} SM_T(X) & := & 0 & & \text{and for } t<T \\ SM_t(X) & := & X_t - \Psi_t^*(X) & \geq & 0. \end{array} \tag{4.4}$$

In the rest of this paper we let Ψ be an assessment according to the Definition 3.1 bounded by identity on $\mathcal{S}_{\mathbb{Q}}$ with $\mathbb{Q} \in \mathcal{M}$ such that its market consistent hull and the supervisory margin exist on $\mathcal{S}_{\mathbb{Q}}$. As a corollary to Proposition 4.3 the supervisory margin has the following properties:

Proposition 4.4.
For $X \in \mathcal{S}_{\mathbb{Q}}$, the supervisory margin $SM(X)$

(i) is independent of hedgeable risks, i.e.

$$SM_t(X) = SM_t\left(X + \sum_{u>t} a_u\cdot\rho(u)\right) \tag{4.5}$$

for all $a = (a_u)_{u\in\mathbb{T}}$,

(ii) satisfies

$$SM_t(X) = -\Psi_t^*\left(\sum_{u>t}\partial X_u\cdot\rho(u)\right) \tag{4.6}$$

(iii) and can be calculated recursively by

$$\begin{array}{rcll} SM_T(X) & = & 0 & \textit{and for } 0\le t<T \\ SM_t(X) & = & -\operatorname*{ess.sup}_{a_{t+1}\in N_{t+1}} \Psi_t\Big((a_{t+1}+\partial X_{t+1})\cdot\rho(t+1) & \\ & & \qquad -SM_{t+1}(X)\cdot\rho(t+1)\,\mathbb{1}_{>t+1}\Big). & \end{array} \tag{4.7}$$

Proof.
(i) follows immediately from Definition 4.2 and (ii) from (3.3) and the cash invariance of Ψ^*.
(iii) By (4.6) and Proposition 4.3 we get for $t < T$

$$\begin{aligned} SM_t(X) &= -\operatorname{ess.sup}_{a_{t+1}} \Psi_t\big(a_{t+1} \cdot \rho(t+1) + \partial X_{t+1} \cdot \rho(t+1) \\ &\qquad\qquad + \Psi^*_{t+1}\big(\sum_{u \geq t+1} \partial X_u \cdot \rho(u)\big) \cdot \rho(t+1)\, \mathbb{I}_{>t+1}\big) \\ &= -\operatorname{ess.sup}_{a_{t+1}} \Psi_t\big((a_{t+1} + \partial X_{t+1}) \cdot \rho(t+1) \\ &\qquad\qquad - SM_{1,t+1}(X) \cdot \rho(t+1)\, \mathbb{I}_{>t+1}\big). \end{aligned}$$

□

The next Section contains two important classes of risk assessments.

5. Examples of risk assessments

We first give an example of a **coherent risk assessment** Ψ: here coherent means that Ψ is super-additive and positively homogenous:
For all $\zeta_t \in L_t^+$, and $X, Y \in L$ it holds that

(i) Positive homogeneity: $\Psi_t(\zeta_t \cdot X) = \zeta_t \cdot \Psi_t(X)$,
(ii) Super-additivity: $\Psi_t(X + Y) \geq \Psi_t(X) + \Psi_t(X)$.

Example 5.1.
For $0 < t \leq T$, let $\mathcal{D}_t$ be a closed convex subset of $(L_t^+)^2$ such that for all $(\zeta_t, \xi_t) \in \mathcal{D}_t$ one has

$$\mathbb{E}\left[\zeta_t + \xi_t \middle| \mathcal{F}_{t-1}\right] = 1 \tag{5.1}$$

and in addition for $t = T$

$$\xi_T \equiv 0. \tag{5.2}$$

Moreover, we suppose that for $0 < t$

$$(\mathcal{D} \cap \partial\mathcal{M})_t := \Big\{(\zeta_t, \xi_t) \in \mathcal{D}_t \Big| \zeta_t + \xi_t \in \partial\mathcal{M}_t\Big\} \neq \emptyset. \tag{5.3}$$

(At first sight the notation $(\mathcal{D} \cap \partial\mathcal{M})_t$ may be misleading since $\mathcal{D}_t \subset (L_t^+)^2$ and $\partial\mathcal{M}_t \subset L_t^+$ and as such $(\mathcal{D} \cap \partial\mathcal{M})_t$ is not a simple intersection of $\mathcal{D}_t$ and $\partial\mathcal{M}_t$; nevertheless the notation $(\mathcal{D} \cap \partial\mathcal{M})_t$ is rather intuitive.)

For a process $X \in L$ we set recursively

$$\begin{aligned} \Psi_T(X) &:= 0 \qquad \text{and for } 0 \leq t < T \\ \Psi_t(X) &:= \operatorname*{ess.inf}_{(\zeta_{t+1}, \xi_{t+1}) \in \mathcal{D}_{t+1}} \mathbb{E}\left[(\zeta_{t+1} \cdot X_{t+1} + \xi_{t+1} \cdot \Psi_{t+1}(X)) \cdot \rho_{t+1,t} \middle| \mathcal{F}_t\right]. \end{aligned} \tag{5.4}$$

It can be checked that $(\Psi_t)_{t \in \mathbb{T}}$ is a coherent risk assessment. We can write Ψ_t in a closed form by

$$\Psi_t(X) = \operatorname*{ess.inf}_{\substack{(\zeta_u, \xi_u) \in \mathcal{D}_u, \\ t < u \leq T}} \mathbb{E}\left[\sum_{t < u \leq T} \left(\prod_{t < v < u} \xi_v\right) \zeta_u \cdot X_u \cdot \rho_{u,t} \middle| \mathcal{F}_t\right] \tag{5.5}$$

where of course $\prod_{v\in\emptyset}\ldots = 1$ and $\sum_{u\in\emptyset}\ldots = 0$.

Moreover, the condition (5.3) together with the m-stability of $\mathcal{M}$ implies that there exists $\overline{\mathbb{Q}} \in \mathcal{M}$ such that for all $0 < t \leq T$ there exists $(\overline{\zeta}_t, \overline{\xi}_t) \in \mathcal{D}_t$ with $\overline{\zeta}_t + \overline{\xi}_t = \partial\overline{\mathbb{Q}}_t$. It suffices to define

$$\overline{\mathbb{Q}}(A) := \mathbb{E}\left[\prod_{u=1}^{T}\left(\overline{\zeta}_u + \overline{\xi}_u\right)\mathbb{1}_A\right] \tag{5.6}$$

for a choice $(\overline{\zeta}_u, \overline{\xi}_u) \in (\mathcal{D}\cap\partial\mathcal{M})_u$ for $1 \leq u \leq T$. We fix $\overline{\mathbb{Q}}$.

Then we can show that the risk assessment Ψ is bounded by identity on the set $\mathcal{S}_{\overline{\mathbb{Q}}}$ of discounted $\overline{\mathbb{Q}}$-supermartingales:

Proposition 5.1.
For all $X \in \mathcal{S}_{\overline{\mathbb{Q}}}$ and $0 \leq t < T$ we have

$$\Psi_t(X) \leq X_t. \tag{5.7}$$

Proof.
We prove (5.7) by backward induction on t: For $t = T-1$ we get by the Bayes' formula

$$\begin{aligned}
\Psi_{T-1}(X) &\leq \mathbb{E}\left[\overline{\zeta}_T \cdot X_T \cdot \rho_{T,T-1}\middle|\mathcal{F}_{T-1}\right] = \frac{\mathbb{E}\left[\frac{\partial\overline{\mathbb{Q}}}{\partial\mathbb{P}} \cdot X_T \cdot \rho_{T,T-1}\middle|\mathcal{F}_{T-1}\right]}{\mathbb{E}\left[\partial\overline{\mathbb{Q}}/\partial\mathbb{P}\middle|\mathcal{F}_{T-1}\right]} \\
&= \mathbb{E}_{\overline{\mathbb{Q}}}\left[X_T \cdot \rho_{T,T-1}\middle|\mathcal{F}_{T-1}\right] \leq X_{T-1}.
\end{aligned}$$

For $t < T-1$ we use (5.4) and the recursion hypothesis to find

$$\begin{aligned}
\Psi_t(X) &\leq \mathbb{E}\left[\left(\overline{\zeta}_{t+1} \cdot X_{t+1} + \overline{\xi}_{t+1} \cdot \Psi_{t+1}(X)\right) \cdot \rho_{t+1,t}\middle|\mathcal{F}_t\right] \leq \mathbb{E}\left[\left(\overline{\zeta}_{t+1} + \overline{\xi}_{t+1}\right) \cdot X_{t+1} \cdot \rho_{t+1,t}\middle|\mathcal{F}_t\right] \\
&= \frac{\mathbb{E}\left[\frac{\partial\overline{\mathbb{Q}}}{\partial\mathbb{P}} \cdot X_{t+1} \cdot \rho_{t+1,t}\middle|\mathcal{F}_t\right]}{\mathbb{E}\left[\partial\overline{\mathbb{Q}}/\partial\mathbb{P}\middle|\mathcal{F}_t\right]} = \mathbb{E}_{\overline{\mathbb{Q}}}\left[X_{t+1} \cdot \rho_{t+1,t}\middle|\mathcal{F}_t\right] \leq X_t.
\end{aligned}$$

This shows that Ψ is bounded by identity on $\mathcal{S}_{\overline{\mathbb{Q}}}$. □

It is now easy to find the market consistent hull Ψ:

Proposition 5.2.
Recursively, the market consistent hull Ψ^* of Ψ is given by:

$$\begin{aligned}
\Psi^*_T(X) &:= 0 \qquad \textit{and for } 0 \leq t < T \\
\Psi^*_t(X) &:= \underset{(\zeta_{t+1},\xi_{t+1})\in(\mathcal{D}\cap\partial\mathcal{M})_{t+1}}{\operatorname{ess.inf}} \mathbb{E}\left[\left(\zeta_{t+1} \cdot X_{t+1} + \xi_{t+1} \cdot \Psi^*_{t+1}(X)\right) \cdot \rho_{t+1,t}\middle|\mathcal{F}_t\right].
\end{aligned} \tag{5.8}$$

or in closed form:

$$\Psi^*_t(X) = \underset{\substack{(\zeta_u,\xi_u)\in(\mathcal{D}\cap\partial\mathcal{M})_u,\\ t<u\leq T}}{\operatorname{ess.inf}} \mathbb{E}\left[\sum_{t<u\leq T}\left(\prod_{t<v<u}\xi_v\right)\zeta_u \cdot X_u \cdot \rho_{u,t}\middle|\mathcal{F}_t\right] \tag{5.9}$$

The proof is similar to the one in Artzner and Eisele (2010) Proposition 2.1 for the one-period case.

A special case is Definition 3.3, the best estimate of final value, where we set for some $\mathbb{Q} \in \mathcal{M}$

$$\mathcal{D}_T = \{(\partial \mathbb{Q}_T, 0)\} \quad \text{and for } 0 < t < T \tag{5.10}$$

$$\mathcal{D}_t = \{(0, \partial \mathbb{Q}_t)\}. \tag{5.11}$$

Example 5.2.

We start with the following definition of a **conditional bounded VaR-operator**:

Definition 5.1.
Let $\mathbb{Q} \ll \mathbb{P}$ and $\overline{\mathbb{Q}} \in \mathcal{M}$ two probabilities and $\alpha = (\alpha_t)_{0 \le t < T} \in \prod_{0 \le t < T} L_t^+$ with $\alpha_t \in [0,1]$ a.s. Now for $0 \le t < T$ and $Y \in L_{t+1}$ we define

$$bVaR_t(Y) := \operatorname{ess.inf}\left\{ Z \in L_t \middle| \, \mathbb{Q}\left[Y \cdot \rho_{t+1,t} \le Z \middle| \mathcal{F}_t\right] > \alpha_t \right\} \wedge \mathbb{E}_{\overline{\mathbb{Q}}}\left[Y \cdot \rho_{t+1,t} \middle| \mathcal{F}_t\right] \tag{5.12}$$

where we suppressed the dependence on $\mathbb{Q}$, $\overline{\mathbb{Q}}$, and α.

Now for $0 < t < T$ let $\mathcal{C}_t$ be closed convex subset of L_t^+ with

$$0 \le \chi_t \le 1 \quad \text{for all } \chi_t \in \mathcal{C}_t. \tag{5.13}$$

We define $\widetilde{\Psi}$ for a process $X \in L$ by

$$\begin{aligned} \widetilde{\Psi}_T(X) &:= 0 \\ \widetilde{\Psi}_{T-1}(X) &:= bVaR_{T-1}(X_T) \qquad \text{and for } 0 \le t < T-1 \\ \widetilde{\Psi}_t(X) &:= \operatorname*{ess.inf}_{\chi_{t+1} \in \mathcal{C}_{t+1}} bVaR_t\left(\chi_{t+1} \cdot X_{t+1} + (1-\chi_{t+1}) \cdot \widetilde{\Psi}_{t+1}(X)\right). \end{aligned} \tag{5.14}$$

We leave it to the reader to verify that $\widetilde{\Psi}$ is a homogeneous risk assessment on L, but obviously not super-additive.

Again, $\widetilde{\Psi}$ is bounded by identity on $\mathcal{S}_{\overline{\mathbb{Q}}}$:

Proposition 5.3.
For all $X \in \mathcal{S}_{\overline{\mathbb{Q}}}$ and $0 \le t < T$ we have

$$\widetilde{\Psi}_t(X) \le X_t. \tag{5.15}$$

Proof.
For $t = T-1$ we have $\widetilde{\Psi}_{T-1}(X) \le \mathbb{E}_{\overline{\mathbb{Q}}}\left[X_T \cdot \rho_{T,T-1} \middle| \mathcal{F}_{T-1}\right] \le X_{T-1}$.
For $0 \le t < T-1$ we have by induction

$$\begin{aligned} \widetilde{\Psi}_t(X) &\le \operatorname*{ess.inf}_{\chi_{t+1} \in \mathcal{C}_{t+1}} \mathbb{E}_{\overline{\mathbb{Q}}}\left[\left(\chi_{t+1} \cdot X_{t+1} + (1-\chi_{t+1}) \cdot \widetilde{\Psi}_{t+1}(X)\right) \cdot \rho_{t+1,t} \middle| \mathcal{F}_t\right] \\ &\le \mathbb{E}_{\overline{\mathbb{Q}}}\left[X_{t+1} \cdot \rho_{t+1,t} \middle| \mathcal{F}_t\right] \le X_t. \end{aligned}$$

However, without further assumptions there does not exist a simple form for the market consistent hull $\widetilde{\Psi}^*$ of $\widetilde{\Psi}$. □

We will now apply the best estimate $\Psi_{\mathbb{Q}}$ and the general risk assessment Ψ to the special situation of a commercial bank.

6. A multiperiod business plan for a bank

To establish a business plan for a bank, we first distinguish between amounts like the sum of the deposit accounts which have a retrospective character, and the future incoming or outgoing stochastic flows.

The bank has signed loan contracts which create an adapted stream of future payments $y = (y_t)_{t\in\mathbb{T}} \in L^+$, $y_0 = 0$. Interest rates and payments on the principal at time t are included in y_t. The risks which lie in the process of loan payments are primordial for the bank. In critical situations the first thing to do is to stop further risky loans. For these reasons, supervisory calculations will be done under a **run-off assumption for loans**, i.e. no new loan contracts are taken in consideration and all payments have to be done before or at the supervisory time horizon T.

By $D = (D_t)_{t\in\mathbb{T}}$ we denote the process of the bank's deposit account. It evolves out of $D_0 \geq 0$ by

$$\partial D = d + z \tag{6.1}$$

where $d = (d_t)_{t\in\mathbb{T}}$, $d_0 = 0$, is the exogenous flow process of new deposits or withdrawals and $z = (z_t)_{t\geq 1}$ denotes the remuneration for the deposit accounts on top of the risk-free remuneration. We assume

$$D \geq 0 \qquad \text{or equivalently} \qquad d_t \geq -D_{t-1} \cdot \rho_{t-1,t} - z_t \tag{6.2}$$

for $0 < t \leq T$.

For example, if the bank itself has obligations on the financial market for which it has to pay an additional spread on top of the growth rate of the eligible asset, then the obligations can be treated as deposits and the flow of extra remunerations z_t should contain these spread rates. An other example would be the situation of a "bank-insurance" where outstanding loans and claims enter into the business plan.

Let $A \in L$ be the process of the bank's current amount of tradeable assets with the initial amount A_0. The process A satisfies the bookkeeping equation

$$\partial A = a + y + d \tag{6.3}$$

where $a = (a_t)_{t\in\mathbb{T}}$, $a_0 = 0$ is the process of the differences of the company's tradeable asset amount — before the payment of loans y_t and the change in the deposits d_t — with respect to an evolution of a mere investment of A in the eligible asset ρ. Notice that the deposit remunerations z_t modify the deposit amount, but are not immediately paid out; therefore they do not appear in (6.3).

Definitions 6.1.

(i) Supervision is done under the assumption that the tradeable asset value process A satisfies the **self-financing condition**

$$a \in N. \tag{6.4}$$

i.e. a is a process of trading-risk exposures.

(ii) The sextuple $B_t := (A_t, D_t, a \cdot \mathbb{1}_{>t}, d \cdot \mathbb{1}_{>t}, y \cdot \mathbb{1}_{>t}, z \cdot \mathbb{1}_{>t})$ satisfying (6.1), (6.3), and (6.4) after t is called a **business plan for the bank** from t on.
In the following we suppress the indicator functions in B_t and write simply

$$B_t = (A_t, D_t, a, d, y, z) \tag{6.5}$$

7. The risk bearing capital

We continue to use the eligible asset ρ as numéraire and to evaluate market consistent prices by a trading-risk neutral probability $\mathbb{Q} \in \mathcal{M}$.

For the process of loan payments y and the process of extra remunerations z we introduce the processes of cumulated past payments as well as market consistent estimations of the future payments.

Definitions 7.1.

(i) The process of **cumulated past loan repayments** Y° is given by

$$Y_t^\circ := \sum_{u=1}^{t} y_u \cdot \rho_{u,t}, \quad t \in \mathbb{T}, \tag{7.1}$$

while the process of **market consistent best estimations of future loan payments** Y is

$$Y_t := \mathbb{E}_{\mathbb{Q}}\left[\sum_{u>t} y_u \cdot \rho_{u,t} \middle| \mathcal{F}_t\right]. \tag{7.2}$$

(ii) The process of **cumulated extra remuneration** Z° is

$$Z_t^\circ := \sum_{u=1}^{t} z_u \cdot \rho_{u,t}, \quad t \in \mathbb{T}, \tag{7.3}$$

while the process of **best estimations of extra remuneration** Z is

$$Z_t := \mathbb{E}_{\mathbb{Q}}\left[\sum_{u>t} z_u \cdot \rho_{u,t} \middle| \mathcal{F}_t\right]. \tag{7.4}$$

(iii) We define the **process of risk bearing capital** as

$$RBC = RBC(B) = A + Y - D - Z. \tag{7.5}$$

Remark 7.1.
In usual bank accounting, the estimated future loan payment Y_t appears as the differences between the nominal values of loans in the bank book and the **provision for expected losses (PEL)**. The later is a market consistent estimation of loan losses which does not include a security margin.

Proposition 7.1.

(i) The process Y has (up to sign) the discounted $\mathbb{Q}$-martingale decomposition

$$Y = \widetilde{Y} - Y^\circ \tag{7.6}$$

where

$$\widetilde{Y}_t := \mathbb{E}_{\mathbb{Q}}\left[\sum_{u\in\mathbb{T}} y_u \cdot \rho_{u,t} \middle| \mathcal{F}_t\right]. \tag{7.7}$$

The increments of $\widetilde{Y}$

$$\partial\widetilde{Y}_t = \mathbb{E}_{\mathbb{Q}}\left[\sum_{u=t}^{T} y_u \cdot \rho_{u,t} \middle| \mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}}\left[\sum_{u=t}^{T} y_u \cdot \rho_{u,t-1} \middle| \mathcal{F}_{t-1}\right] \cdot \rho_{t-1,t} \tag{7.8}$$

provide immediately the property of a discounted $\mathbb{Q}$-martingale:

$$\mathbb{E}_{\mathbb{Q}}\left[\partial\widetilde{Y}_t \cdot \rho_{t,t-1} \mid \mathcal{F}_{t-1}\right] = 0. \tag{7.9}$$

(ii) Similarly, Z has the discounted $\mathbb{Q}$-martingale decomposition

$$Z = \widetilde{Z} - Z^{\circ} \tag{7.10}$$

where

$$\widetilde{Z}_t := \mathbb{E}_{\mathbb{Q}}\left[\sum_{u\in\mathbb{T}} z_u \cdot \rho_{u,t}\Big|\mathcal{F}_t\right]. \tag{7.11}$$

The increments of $\widetilde{Z}$ are

$$\partial\widetilde{Z}_t = \mathbb{E}_{\mathbb{Q}}\left[\sum_{u=t}^{T} z_u\rho_{u,t}\Big|\mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}}\left[\sum_{u=t}^{T} z_u \cdot \rho_{u,t-1}\Big|\mathcal{F}_{t-1}\right]\cdot\rho_{t-1,t}. \tag{7.12}$$

(iii) Moreover, the increments of the risk bearing capital have the form

$$\partial RBC = a + \partial\widetilde{Y} - \partial\widetilde{Z} \tag{7.13}$$

so that RBC is also a discounted $\mathbb{Q}$-martingale.

Proof.
Part (i) of the Proposition is obvious. To show (7.13), we notice that $\partial RBC_t = \partial(A-D)_t + \partial Y_t - \partial Z_t = a_t + y_t - z_t + \partial Y_t - \partial Z_t = a_t + \partial(Y^{\circ} + Y)_t - \partial(Z^{\circ} + Z)_t = a_t + \partial\widetilde{Y}_t - \partial\widetilde{Z}.$ ☐

Corollary 7.2.
As an immediate consequence of the cash invariance and (7.13) we get for a business plan $B_t = (A_t, D_t, y, z, d, a)$

$$\Psi_t(RBC(B)) = A_t + Y_t - D_t - Z_t + \Psi_t\left(\sum_{u>t}\left(a_u + \partial\widetilde{Y}_u - \partial\widetilde{Z}_u\right)\cdot\rho(u)\right), \tag{7.14}$$

$$\Psi_t^*(RBC(B)) = A_t + Y_t - D_t - Z_t + \Psi_t^*\left(\sum_{u>t}\left(\partial\widetilde{Y}_u - \partial\widetilde{Z}_u\right)\cdot\rho(u)\right),. \tag{7.15}$$

The supervisory margin $SM_t(B)$ of a business plan $B_t = (A_t, D_t, y, z, d, a)$ is defined as the one of its risk bearing capital $RBC(B)$. According to Definition 4.2 we find by (4.4) and (7.13)

$$SM_t(B) := SM_t(RBC(B)) = -\Psi_t^*\left(\sum_{u>t}(\partial\widetilde{Y}_u - \partial\widetilde{Z}_u)\cdot\rho(u)\right). \tag{7.16}$$

Remark 7.2.
The last equation shows that the supervisory margin $SM_t(B)$ indeed depends only on the in general non-hedgeable flow processes y and z; it is independent of the amounts of assets A_t and deposits D_t, and of the flows of trading risk exposures a and deposits d.

From Proposition 4.4 (i) we know that the supervisory margin measures only non-hedgeable risks.

8. Solvability and acceptability

Supervision of a bank's business plan starts with a multiperiodic risk assessment Ψ applied to its risk bearing capital.

Definitions 8.1.

(i) The business plan $B = B_t = (A_t, D_t, a^*, d, y, z)$ has a **t-optimal process of trading risk exposures** $a^* \in N$ if a^* is t-optimal for the process $X = RBC(B)$ of risk bearing capital according to Definition 4.1.

(ii) The business plan B is called **t-solvable** if

$$\Psi_t^*\,(RBC(B)) \geq 0. \tag{8.1}$$

(iii) A business plan B is **t-acceptable** if

$$\Psi_t\,(RBC(B)) \geq 0. \tag{8.2}$$

(iv) The **free equity capital of the bank's business plan at date** t is

$$F_t(B) := \Psi_t(RBC(B)). \tag{8.3}$$

By the cash invariance of the risk assessment, it can be disposed of — if positive — without loosing acceptability.

The existence of an optimal process of trading risk exposures opens the possibility for t-solvable banks to become acceptable by a change of its portfolio.

Theorem 8.1 (Acceptability by rebalancing)**.**
Suppose that optimal replicating portfolios exist.
For $t < T$, a business plan $B_t = (A_t, D_t, a, d, y, z)$ can be made t-acceptable by changing its trading risk exposures a (rebalancing) if and only if B_t is t-solvable.

Proof.
Let a_1^* be a process of t-optimal trading risk exposures for the business plan $B_t = (A_t, D_t, a, d, y, z)$. and define the business plan $B_t^* = (A_t, D_t, a^*, d, y, z)$. The t-solvable condition (8.1) of B_t implies

$$0 \leq \Psi_t^*(RBC(B)) = F_t(B^*).$$

The business plan B_t^* is t-acceptable.
Conversely, let $\widetilde{a}$ be a rebalancing of the trading risk exposures of $B_t = (A_t, D_t, a, d, y, z)$ such that $\widetilde{B}_t = (A_t, D_t, \widetilde{a}, d, y, z)$ is t-acceptable. Then by (7.2) and (7.15)

$$\begin{aligned} 0 \quad &\leq \quad F_t(\widetilde{B}) = A_t + Y_t - D_t - Z_t + \Psi_t\left(\sum_{u>t}(\widetilde{a}_u + \partial\widetilde{Y}_u - \partial\widetilde{Z}_u)\cdot\rho(u)\right) \\ &\leq \quad A_t + Y_t - D_t - Z_t + \Psi_t^*\left(\sum_{u>t}(\partial\widetilde{Y}_u - \partial\widetilde{Z}_u)\cdot\rho(u)\right) = \Psi_t^*\,(RBC(B)) \end{aligned}$$

which means that B_t is t-solvable. $\square$

9. Regulated transfer and the cost-of-capital method

Even if a bank's business plan is not solvable and therefore by the last theorem can not be made acceptable by rebalancing of its tradeable asset portfolio, there is still — under some conditions — a possibility to protect the depositors, namely by a *regulated* transfer to a new bank; regulated means that after the transfer of loans and deposits both the new bank and the old one must be acceptable. This can be reached by the fact that the shareholders of the new bank capitalize it by new own funds. But they are only willing to do so if they can expect some additional profit out of their engagement. The value of the possible profit (neglecting limited liabilities) is the **risk margin** associated to loans and deposits. One can describe it as the cost to attract the new capital. Therefore the transfer principle is also called the **cost-of-capital method**.

Simultaneously, the cost-of-capital method delivers a distinction between **external capital**[3] (also characterized as **"Fremd"-capital**) and **equity capital** (or **"Eigen"-capital**) on the passive side of the supervisory balance sheet.

Before deriving the equations for the risk margin and the equity capital requirement for non-hedgeable risk, we give a general description of the transfer principle (see also Eisele and Artzner (2011) Section 6). To simplify the following considerations we set $t = 0$, but the method is valid for any $t < T$.
The thought-of transfer is standardized by some conditions: First, the transfer object is the amount of deposits D_0 and the flows of deposits d, loans y and extra-remunerations z. Its market consistent value is $V_0 := Y_0 - D_0 - Z_0$. Second, the new bank is assumed — after the transfer of the future flow processes d, y, and z — to apply a business plan $B_0^* = (A_0^*, D_0, a^*, d, y, z)$ with a process of 0-optimal trading risk exposures a^* and zero free capital, i.e.

$$
\begin{aligned}
0 &= F_0(B^*) \qquad\qquad \text{or equivalently} \\
A_0^* &= -Y_0 + D_0 + Z_0 - \Psi_0^*\left(\sum_{u>0}(\partial\widetilde{Y}_u - \partial\widetilde{Z}_u)\cdot\rho(u)\right) \\
&= -V_0 + SM_0(B^*).
\end{aligned}
\tag{9.1}
$$

We call B_0^* the **reference business plan**[4] which is 0-acceptable. Note that here A_0^* may also be negative, in which case it designs debts the new bank takes on the financial market.

For simplicity, assume $V_0 = SM_0(B^*) - A_0^* \geq 0$. This value serves as benchmark in the negotiations between the transferor bank and the new one.
Obviously, the new bank is not willing to pay V_0 as price for the whole transfer since it does not contain a higher return on the risky investment than the risk-free eligible asset. Therefore, the price W_0 the new bank is willing to pay will be less than V_0: $W_0 < V_0$. The difference $V_0 - W_0$ is the **reduction** the transferor has to make in order to attract new capital.

On the other hand, looking at the end of the supervisory horizon T, where all risky positions are dissolved, the new bank can hope for the leftover

$$
\left((A_0^* - D_0)\rho_T + \sum_{u>0}(a_u^* + y_u - z_u)\cdot\rho_{u,T}\right)^+ . \tag{9.2}
$$

This is the *call option of the regulated transfer*. Neglecting the limited liability in (9.2) expressed by the $(\ldots)^+$-sign, the market consistent value of the underlying is exactly $A_0^* + V_0 = SM_0(B^*) \geq 0$.

[3] We prefer the term "external capital", proposed to us by Hans Gerber, to the notion of debts capital, since there are cases where these "debts" are due to nobody, but serve purely as prudence capital for adverse situations.

[4] The terminology of a reference business plan stems from the Solvency II project where the new company is called "reference undertaking".

Therefore, the price paid W_0 will be greater that $-A_0^*$: the difference $W_0 + A_0^* > 0$ is the price of the call option of the regulated transfer. This call option price $W_0 + A_0^*$ is **hard equity capital**: it cannot be replaced by deposits or obligations of the bank and therefore has to be collected from the shareholders. It will be *locked* into the company which remains subject to supervision over time. In other words: The quantity $W_0 + A_0^*$ is a lower bound on the shareholders' contributed money.

The reduction $V_0 - W_0$ appears as external capital on the passive side of the balance sheet. It is also the expected earning for the new bank if we neglect the limited liability as expressed in (9.2).

According to the cost-of-capital method the quotient of the reduction $V_0 - W_0$ by the hard equity capital $W_0 + A_0^*$ is called the **cost-of-capital ratio** q_0:

$$q_0 := \frac{V_0 - W_0}{W_0 + A_0^*}. \tag{9.3}$$

The ratio q_0 is the price the transferor has to pay for one unit of hard equity capital invested in the bank.

Since by (9.1) $SM_0(B^*) = V_0 + A_0^* = (V_0 - W_0) + (W_0 + A_0^*) = (1 + 1/q_0)(V_0 - W_0)$, the external capital part $V_0 - W_0$ resp. the hard equity capital part $W_0 + A_0^*$ of the solvency margin $SM_0(B^*)$ satisfy the equalities:

$$\begin{aligned} V_0 - W_0 &= \frac{q_0}{1+q_0} SM_0(B^*), \qquad \text{respectively} \\ W_0 + A_0^* &= \frac{1}{1+q_0} SM_0(B^*). \end{aligned} \tag{9.4}$$

By condition (9.1) the new bank is acceptable. For the old bank which had initially the current asset A_0, got rid of deposits and the flows d, y, and z, and moreover received the transaction price W_0 for it, the acceptability condition is simply:

$$\begin{aligned} 0 \leq A_0 + W_0 &= A_0 + V_0 - (V_0 - W_0) \qquad \text{or equivalently} \\ A_0 + Y_0 - D_0 - Z_0 &\geq \frac{q_0}{1+q_0} SM_0(B^*). \end{aligned} \tag{9.5}$$

This means that the bank's market consistent value must be greater than the external capital part $\frac{q_0}{1+q_0} SM_0(B^*)$ of the supervisory margin.

10. Risk margin and the non-hedgeable equity capital requirement

Coming back to the general case $t < T$, we define a sequence of transfer ratios:

Assumption 10.1.
*For $t < T$, the **cost-of-capital ratio***

$$q_t \in \overline{L}_t^+ \tag{10.1}$$

is fixed by the supervisor for a regulated and standardized transfer of the deposit D_t, the flow processes of deposits d, of loan payments y, and of remunerations z. For completeness, we add

$$q_T = 0. \tag{10.2}$$

The external and the equity capital parts in (9.4) lead to the following definitions:

Definition 10.1.
Let $t \in \mathbb{T}$ and $B_t = (A_t, D_t, a, d, y, z)$ a business plan.

(i) The **risk margin** $RM_t(B)$ is the external capital part of the supervisory margin $SM_t(B)$:

(10.3) $$RM_t(B) := \frac{q_t}{1+q_t}\, SM_t(B) \geq 0.$$

(ii) The **non-hedgeable equity capital requirement (**ECR^***)** is the hard equity capital part of the supervisory margin $SM_t(B)$:

(10.4) $$ECR_t^*(B) := \frac{1}{1+q_t}\, SM_t(B) \geq 0.$$

(iii) The **hedgeable equity capital requirement (**$ECR^h(B)$**)** is defined as

(10.5) $$ECR_t^h(B) := \Psi_t^*(RBC(B)) - \Psi_t(RBC(B)) \geq 0.$$

Remark 10.1.

(i) It is worthwhile noticing that both the risk margin RM and the non-hedgeable equity capital requirement (ECR^*) are defined via the supervisory margin. Consequently, they both depend only on the flow processes of loans y and of extra remunerations z (see Remark 7.2). This justifies in particular the adjective "non-hedgeable" for ECR^*.

(ii) A look to the equations (8.3) and (4.1) shows: A business plan $B_t = (A_t, D_t, a, d, y, z)$ has a t-optimal process of trading risk exposures if and only if its hedgeable equity capital requirement $ECR^h(B)$ vanishes: $ECR^h(B) = 0$.

With the above definitions, the condition (9.5) can be rephrased in the general case as

(10.6) $$A_t + Y_t - D_t - Z_t \geq RM_t(B).$$

This is the **transferability condition**.

For a business plan $B_t = (A_t, D_t, a, d, y, z)$ we have characterized the following supervisory accounting items:

- on the active side of the balance sheet: $A_t + Y_t$,
- on the passive side as external capital: $D_t + Z_t + RM_t(B)$,
- and the non-hedgeable and the hedgeable equity capital requirements: $ECR_t^*(B) + ECR_t^h(B)$
  These are locked-in amounts of equity capital.
- Finally, there is the free equity capital: $F_t(B)$.

Summing up, we get the **supervisory accounting equality**:

Proposition 10.1.

(10.7) $$A_t + Y_t = D_t + Z_t + RM_t(B) + ECR_t^*(B) + ECR_t^h(B) + F_t(B),$$

Proof.
We use successively the equations (8.3), (10.5), (7.15), (7.16), (10.3), and (10.4) to get

$$F_t(B) + ECR_t^h(B) = \Psi_t^*(RBC(B) = A_t + Y_t - D_t - Z_t - RM_t(B) - ECR_t^*(B).$$

This is equivalent to (10.7). □

The **equity capital (EC)**, sometimes also called *net asset value (NAV)*, of a business plan $B_t = (A_t, D_t, a, d, y, z)$ is the difference between the active part of the balance sheet minus the external capital:

$$EC_t(B) := A_t - D_t + Y_t - Z_t - RM_t(B). \tag{10.8}$$

If we compare the equity capital with different levels of the equity capital requirement we find the following necessary and sufficient conditions for the **hierarchy of the supervisor's interferences:**

- $EC_t(B) \geq ECR_t^*(B) + ECR_t^h(B))$ $\longleftrightarrow$ acceptability,
  (equivalent to $\Psi_t(RBC(B)) \geq 0$),
- $EC_t(B) \geq ECR_t^*(B)$ $\longleftrightarrow$ acceptability after rebalancing,
  (equivalent to $\Psi_t^*(RBC(B)) \geq 0$),
- $EC_t(B) \geq 0$ $\longleftrightarrow$ transferability,
- $EC_t(B) < 0$ $\longleftrightarrow$ bankruptcy.

11. A simple allocation principle

So far we have analyzed a model of a bank, presented by a bank's business plan $B_t = (A_t, D_t, a, d, y, z)$. It incorporates four stochastic flow processes: the flow a of trading risk exposures, the future flow d of deposits, the process of loan repayments y, and the remuneration process z. From the supervisor's point of view the four flows have the following impacts:

(i) As long as the future deposits are remunerated by market rates, more precisely by the change in the eligible asset, the flow d of deposits can be completely hedged. Since the eligible asset is risk free — subjectively to the supervisor's view — the future deposits can be installed without any risk. This shows up in the risk assessment of the process of risk bearing capital, where in the formulas (7.14) and (7.15) the flow process d does not appear. Consequently, the supervisory accounting items: the supervisory margin $SM(B)$ including the risk margin $RM(B)$ and the non-hedgeable equity capital requirement $ECR^*(B)$, as well as the hedgeable equity capital requirement $ECR^h(B)$ do not depend on d (see also the Remarks 7.2 and 10.1).

(ii) The supervisory margin is based on the idea of an optimal replicating portfolio a^*. It is also independent of the process a of trading risk exposures. The flow of trading risk exposures enters only in the hedgeable equity capital requirement $ECR^h(B)$ given by (10.5) in connection with (7.14) and (7.15). The question to manage the hedgeable equity capital requirement $ECR^h(B)$ is thus separated from the assessment of the other risks hidden in the processes of loan payments y and remunerations z in so far as the $ERC^h(B)$ can always reduced to zero choosing an optimal replicating portfolio (see Remark 10.1 (ii)).

(iii) The two remaining flow processes y and z are mingled together in the supervisory margin $SM(B)$ and therefore also in the risk margin and in the non-hedgeable equity capital requirement.

In our case the **allocation problem** presents itself as the question how to attribute accounting items like the two margins or more important the non-hedgeable equity capital requirement to the two flow processes y and z individually in such a way that their sums equals the global items. We give here a rather simple allocation principle which however has the advantage not to impose additional properties on our risk assessment Ψ. Remember that we did impose neither a concavity or super-additivity nor coherence condition on Ψ. (For more involved allocation principles the reader is referred to Delbaen (2011) Section 9 or Tasche (2006).)

We regard the supervisory margins for the processes y and z separately. In analogy to equation (7.16) we define

$$
\begin{aligned}
SM_t(y) &:= -\Psi^* \left(\sum_{u>t} \partial \widetilde{Y}_u \cdot \rho(u) \right) \geq 0, \\
SM_t(z) &:= -\Psi^* \left(-\sum_{u>t} \partial \widetilde{Z}_u \cdot \rho(u) \right) \geq 0.
\end{aligned} \tag{11.1}
$$

Then we apply a simple thumb rule for the **allocation of supervisory margin (ASM)** to y, resp. z:

$$
\begin{aligned}
ASM_t(y) &:= \frac{SM_t(y)}{SM_t(y) + SM_t(z)} \cdot SM_t(B), \\
ASM_t(z) &:= \frac{SM_t(z)}{SM_t(y) + SM_t(z)} \cdot SM_t(B).
\end{aligned} \tag{11.2}
$$

Using (10.3) and (10.4) the last relation is also kept for the **allocation of the risk margin** (ARM) and the **allocation of the non-hedgeable equity capital requirement** ($AECR^*$) to y, resp. z:

$$
\begin{aligned}
ARM_t(y) &:= \frac{q_t}{1+q_t} \cdot ASM_t(y), \\
ARM_t(z) &:= \frac{q_t}{1+q_t} \cdot ASM_t(z),
\end{aligned} \tag{11.3}
$$

$$
\begin{aligned}
AECR^*_t(y) &:= \frac{1}{1+q_t} \cdot ASM_t(y), \\
AECR^*_t(z) &:= \frac{1}{1+q_t} \cdot ASM_t(z).
\end{aligned} \tag{11.4}
$$

Here the same cost-of-capital ratio should be applied as in (10.4) and (10.5). In particular the allocation of the non-hedgeable equity capital requirement can be used to calculate the return on equity capital with respect to the loans y or the commitments z.

12. A transfer restricted to outstanding loans or to deposits with extra remunerations

The supervisory margin for outstanding loans and the one for deposits with extra remunerations were given in (11.1). Like in (10.3) and (10.4), the corresponding risk margins and non-hedgeable equity capital requirements are

$$
RM_t(y) = \frac{q_t(y)}{1+q_t(y)} SM_t(y) \quad \text{and} \quad ECR^*_t(y) = \frac{1}{1+q_t(y)} SM_t(y) \tag{12.1}
$$

$$
RM_t(z) = \frac{q_t(z)}{1+q_t(z)} SM_t(z) \quad \text{and} \quad ECR^*_t(z) = \frac{1}{1+q_t(z)} SM_t(z) \tag{12.2}
$$

where however the cost-of-capital ratios may depend on the flows y, resp. z.

Let's first assume that the outstanding loans $y \cdot \mathbb{I}_{>t}$ are transferred to a reference bank. The transferor receives the regulated transfer price $Y_t - RM_t(y)$. After transfer it has business plan $B_{1,t} = (A_t + Y_t - RM_t(y), D_t, a, d, 0, z)$ and the equity capital

$$EC_t(B_1) = A_t - D_t + Y_t - Z_t - RM_t(y) - RM_t(z).$$

Therefore, the conditions for a transfer of $y \cdot \mathbb{I}_{>t}$ are:

- $EC_t(B_1) \geq ECR^*_t(z) + ECR^h_t(B_1)$ $\longleftrightarrow$ transferability of y
  (equivalent to $\Psi_t(RBC(B_1)) \geq 0$) keeping the trading risk exposures a,

- $EC_t(B_1) \geq ECR^*_t(z)$ $\longleftrightarrow$ transferability of y
  (equivalent to $\Psi^*_t(RBC(B_1)) \geq 0$) under optimal trading risk exposures a^*_1.

When the deposits D_t and the flow $z \cdot \mathbb{I}_{>t}$ of future extra remunerations are transferred, the transferor has to pay the regulated price $D_t + Z_t + RM_t(z)$ and thereafter has the business plan $B_{2,t} = (A_t - D_t - Z_t - RM_t(z), 0, a, 0, y, 0)$. Its equity capital is the same as before

$$EC_t(B_2) = A_t - D_t + Y_t - Z_t - RM_t(y) - RM_t(z) = EC_t(B_1).$$

The conditions for a transfer of D_t and $z \cdot \mathbb{I}_{>t}$ are:

- $EC_t(B_2) \geq ECR^*_t(y) + ECR^h_t(B_2)$ $\longleftrightarrow$ transferability of D_t and z
  (equivalent to $\Psi_t(RBC(B_2)) \geq 0$) keeping the trading risk exposures a,

- $EC_t(B_2) \geq ECR^*_t(y)$ $\longleftrightarrow$ transferability of y
  (equivalent to $\Psi^*_t(RBC(B_2)) \geq 0$) under optimal trading risk exposures a^*_2.

13. Conclusion

As we hopefully have made clear, the condition of absence of supervisory arbitrage is fundamental: it is evidently a burden imposed on the risk assessments to be regarded, but on the other hand it helps to define the market consistent hull of a risk assessment and thereafter the supervisory margin.

The supervisory margin is a very important tool since it allows to cover simultaneously the non-hedgeable risks on the active and on the passive side of the balance sheet. This is of special importance for banks where loan risks (active) and commitment risks (passive) come hand-in-hand.

Since the supervisory margin concentrates on non-hedgeable risks, it leaves the hedgeable ones to be captured by the hedgeable equity capital requirement.

With these both general and technical considerations we seriously hope to contribute to the ongoing and important discussion on bank supervision, in particular with respect to Basel III. Moreover, the identical fondations between the present work and the one about insurance supervision in Eisele and Artzner (2011) should encourage the tendency to similar supervisory principles in banking and insurance.

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